Generalization of Black-Scholes-Merton model and risk minimizing pricing

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IISER PUNE

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   - Black-Scholes-Merton Model
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What is a European Option?

A European call option is a contract with the following conditions:

At a prescribed time in future (expiration date T) person A (the holder of the option) has right to buy the stock (S_T) for a prescribed price (strike price K) from person B (the writer of the option).

Two Central Issues

The price or value of the option at maturity is

$$C_T = (S_T - K)^+ \;=\; \max(S_T - K, 0).$$

At the time of writing the option, S_T is unknown. Two problems :

(i) How much the holder should pay to the writer at time $t = 0$ for an asset worth $(S_T - K)^+$ at time T? This is the problem of pricing the option.

(ii) How should the writer, who earns the premium initially, generate an amount $(S_T - K)^+$ at time T? This is the problem of hedging the option.

Heuristics

Let H_t be the present price of an asset worth H at time T.
Let $B_t = e^{rt}$, and $r(> 0)$ is bank rate.

- If H is deterministic then $H_t = B_t \frac{H}{B_T}$.
- If H is $\mathcal{F}_T$ measurable, then $H_t = Average(B_t \frac{H}{B_T})$?
- Consider a weighted average E^*(irrespective of H), i.e., expectation w.r.t. an equivalent probability measure P^*.
- A necessary condition (NA): **Present price of $S_T = S_t$.** i.e., $E^*[B_t \frac{S_T}{B_T} \mid \mathcal{F}_t] = S_t \Rightarrow \{\frac{S_t}{B_t}\}_t$ is martingale under P^*.
- Thus $H_t = E^*[B_t \frac{H}{B_T} \mid \mathcal{F}_t]$, where E^* is the conditional expectation w.r.t an equivalent martingale measure P^*.
- But P^* may not exist in general. Even if it exists, may not be unique.

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Option Price as a Function of Current Stock

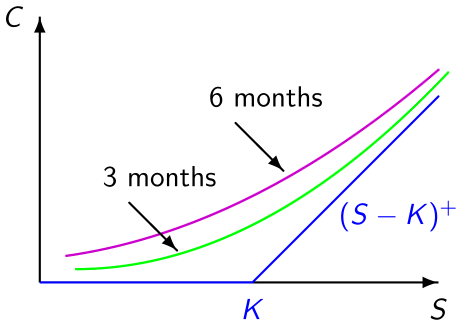

Figure : Call option prices at time to maturity

Black-Scholes-Merton Model

Market: Two assets

- Money market account
  Risk free interest rate $= r$.
  Present(at time t) worth of initial investment of 1 unit $= B_t$.

$$B_t = e^{rt}, \; r > 0$$
$$dB_t = rB_t dt, B_0 = 1.$$

- One Stock: S_t
  Follows a GBM i.e.

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \; S_0 > 0$$
$$S_t = S_0 \exp((\mu - \frac{1}{2}\sigma^2)t + \sigma W_t)$$

generally $\mu > r$. $\{W_t\}$ is a standard Wiener process. Let $(\Omega, \mathcal{F}, P)$ be the underlying complete probability space and $\mathcal{F}_t$ a right continuous usual filtration generated by S_t.

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Existence of Unique EMM

Under market probability P

- B_t grows at the rate r
- Expected growth rate of S_t is μ.

In order to change P to P^* (equivalent to P) under which both B_t and S_t grow at the same rate, set

$$\frac{dP^*}{dP} := \exp\left(-\int_0^T \frac{\mu - r}{\sigma} dW_t - \frac{1}{2}\int_0^T \left(\frac{\mu - r}{\sigma}\right)^2 dt\right).$$

Then $P \equiv P^*$. Set

$$W_t^* = W_t + \frac{\mu - r}{\sigma} t.$$

Under P^*, W_t^* is a standard Brownian motion. Also S_t satisfies

$$dS_t = rS_t dt + \sigma S_t dW_t^*.$$

Thus under P^* the growth rates of B_t and S_t are the same.
Let

$$S_t^* := e^{-rt} S_t$$

then

$$dS_t^* = \sigma S_t^* dW_t^*,$$

so, $\{S_t^*\}_{t \geq 0}$ is a martingale under P^* i.e., P^* is an EMM.
Existence of an EMM implies that the market is **arbitrage free** under admissible strategies. Again, there is no EMM other than P^*. Thus the unique arbitrage free price of call option at time t is

$$E^*[e^{-r(T-t)}(S_T - K)^+ \mid \mathcal{F}_t].$$

The uniqueness of EMM gives the completeness of the market, i.e. **every contingent claim is attainable by a self financing strategy**.

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Black-Scholes-Merton PDE

Owing to the Markovity of S_t, we define

$$\eta(t,s) := E^*\left[e^{-r(T-t)}(S_T-K)^+ \mid S_t = s\right] = \text{price}.$$

The function η satisfies the following parabolic PDE

$$\frac{\partial}{\partial t}\eta(t,s) + rs\frac{\partial}{\partial s}\eta(t,s) + \frac{1}{2}\sigma^2 s^2\frac{\partial^2}{\partial s^2}\eta(t,s) - r\eta(t,s) = 0$$

$(t,s) \in (0,T) \times \mathbb{R}_+$ with the boundary conditions

$$\eta(T,s) = (s-K)^+, s \in [0,\infty) \qquad \text{and } \eta(t,0) = 0, t \in [0,T].$$

The above equation is referred to as the B-S-M PDE. Solution:

$$\eta(t,s) = s\Phi\left(\frac{\log\frac{s}{K} + (r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}\right)$$
$$-Ke^{-r(T-t)}\Phi\left(\frac{\log\frac{s}{K} + (r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}\right).$$

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Consider a strategy $\pi = (\pi_t^0, \pi_t^1)$, where π_t^0 is the number of units invested in money market account and π_t^1 is the number of units invested in stock at time t. Corresponding to an admissible self-financing strategy π, discounted value of portfolio at time t is given by

$$V_t^*(\pi) := \pi_t^0 + \pi_t^1 S_t^*$$

which also satisfies

$$V_t^*(\pi) = V_0(\pi) + \int_0^t \pi_u^1 dS_u^*.$$

Again, Itô's formula gives, after some manupulation

$$\eta^*(t, S_t) = \eta(0, S_0) + \int_0^t \frac{\partial}{\partial s}\eta(u, S_u) dS_u^*.$$

By comparing both self-financing hedging strategy for call option is

$$\begin{aligned} \pi_t^1 &:= \frac{\partial}{\partial s}\eta(t, S_t) \\ \pi_t^0 &:= \eta^*(t, S_t) - \pi_t^1 S_t^*. \end{aligned}$$

Drawbacks of Black-Scholes-Merton Model

In Black-Scholes-Merton model it is assumed that

- The volatility is constant.
- The interest rate is fixed.

These assumptions do not match reality. Several alternative models are proposed in the literature:

- Stochastic interest rate
- Various stochastic volatility models
- Jump diffusion models etc.

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Markov Modulated Market

Keeping the drawbacks of Black-Scholes-Merton model in mind, we model the floating interest rate r and volatility σ as Markov processes (non-constant w.r.t time).

- $\{X_t\}_{t\geq 0}$: an observable Markov process taking values in $\mathcal{X} = \{1, 2, \ldots, k\}$ with rate matrix $\Lambda = (\lambda_{ij})$.
- X_t is modeled to present hypothetical states of the market at time t.
- Let $r(X_t)$ and $\sigma(X_t)$ be the floating interest rate and market volatility at time t.

This market has NA (No Arbitrage) but is incomplete. Thus

- There are multiple arbitrage free prices of an option.
- Every contingent claim may not be attainable by a self-financing admissible strategy.

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Model Description

One money market account B_t and one stock S_t.

$$\begin{aligned} B_t &= e^{\int_0^t r(X_u)du} \\ dB_t &= r(X_t)B_t dt, B_0 = 1 \\ dS_t &= S_t\ (\mu(X_t)dt + \sigma(X_t)dW_t), S_0 > 0. \end{aligned}$$

$\{X_t\}$ and $\{W_t\}$ are independent. Let $\mathcal{F}_t$ be the right continuous usual filtration generated by X_t and S_t.

REMARK

The Markov modulated market is not complete, the claim H may not be attained through a self financing strategy. An extra cash flow is required to replicate H at the maturity time. Therefore, to price an option the writer must take care of the risk associated with a hedging strategy. Hence he/she would like to seek an "optimal hedging strategy", causing 'minimum' cash flow and a "risk minimizing" price corresponding to that.

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Minimal Martingale Measure

For $i \in \mathcal{X}$, let $\gamma(i) := \frac{\mu(i)-r(i)}{\sigma(i)}$, i.e., $\gamma(i)$ is the **market price of risk** in regime (or mode) i. Let

$$\begin{aligned} \rho_t &:= \exp\left(- \int_0^t \gamma(X_u) dW_u - \frac{1}{2} \int_0^t \gamma(X_u)^2 du \right), \qquad t \leq T \\ dP^* &:= \rho_T \; dP \end{aligned}$$

Note that P^*, as defined above is a probability measure and is equivalent to P. This P^* has some special properties, viz., it is the minimal martingale measure.

Definition

An EMM $P' \equiv P$ is said to be **minimal** if $P' = P$ on $\mathcal{F}_0$ and if any square integrable P- martingale which is orthogonal to martingale part of S_t under P remains a martingale under P'.

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Additional Cash Flow and Quadratic Residual Risk

For a given admissible strategy π let us define

$$
\begin{aligned}
V_t(\pi) &= \pi_t^0 B_t + \pi_t^1 S_t \\
V_t^*(\pi) &= \frac{V_t(\pi)}{B_t} \\
C_t^*(\pi) &:= V_t^*(\pi) - \int_0^t \pi_u^1 dS_u^*, \qquad 0 \le t \le T, \\
R_t(\pi) &:= E[(C_T^*(\pi) - C_t^*(\pi))^2 \mid \mathcal{F}_t].
\end{aligned}
$$

Definition

An admissible strategy π^* is said to be **optimal** (i.e., locally risk minimizing) if the corresponding discounted cost $\{C_t^*(\pi^*)\}$ is a square integrable martingale orthogonal to

$$M_t := \int_0^t \sigma(X_u) S_u^* dW_u.$$

Let $H := (S_T - K)^+$, $H^* = B_T^{-1} H$.

The existence of an optimal strategy for $H \equiv$ The existence of **Föllmer-Schweizer decomposition** of H^* in the form

$$H^* = H_0 + \int_0^T \xi_u^{H^*} \, dS_u^* \, + L_T^{H^*} \qquad \text{under } P. \tag{1}$$

$H_0 \in L^2(\Omega, \mathcal{F}_0, \mathcal{P})$,
$\xi^{H^*} = \{\xi_t^{H^*}\}$ satisfies the square integrability condition,
and $L^{H^*} = \{L_t^{H^*}, 0 \leq t \leq T\}$ is a zero-mean square integrable martingale orthogonal to $\{M_t,\ 0 \leq t \leq T\}$.

For the decomposition (1), the associated optimal strategy $\pi = (\pi_t^0, \pi_t^1)$ is given by

$$\pi_t^1 := \xi_t^{H^*}, \quad \pi_t^0 := V_t^* - \pi_t^1 S_t^*.$$

with

$$V_t^* := H_0 + \int_0^t \xi_u^{H^*} \, dS_u^* \, + L_t^{H^*}, \qquad 0 \leq t \leq T.$$

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with

$$V_t^* := H_0 + \int_0^t \xi_u^{H^*} dS_u^* + L_t^{H^*}, \qquad 0 \le t \le T.$$

Thus the discounted optimal cost $C_t^*(\pi)$ is given by

$$C_t^*(\pi) = H_0 + L_t^{H^*}.$$

The locally risk minimizing price of call option is given by

$$\begin{aligned} & B_t V_t^* \\ &= E^*[B_t H^* \mid \mathcal{F}_t] \\ &= E^*[e^{-\int_t^T r(X_u)du}(S_T - K)^+ \mid \mathcal{F}_t] \\ &= E^*[e^{-\int_t^T r(X_u)du}(S_T - K)^+ \mid S_t, X_t]. \end{aligned}$$

Call Option in Markov Modulated Market

Consider the following equation which is the generalization of Black-Scholes-Merton partial differential equation for the Markov modulated market.

$$\frac{\partial}{\partial t}\varphi(t,s,i) + r(i)s\frac{\partial}{\partial s}\varphi(t,s,i) + \frac{1}{2}\sigma^2(i)s^2\frac{\partial^2}{\partial s^2}\varphi(t,s,i) + \sum_j \lambda_{ij}\varphi(t,s,j) = r(i)\ \varphi(t,s,i) \quad (2)$$

$(t,s,i) \in \mathcal{D} := (0,T) \times \mathbb{R}_+ \times \mathcal{X}$ with the boundary conditions $\varphi(t,0,i) = 0$ for all $t \in [0,T]$ and

$$\varphi(T,s,i) = (s-K)^+; \qquad s \in [0,\infty); \qquad i = 1,2,\ldots,k. \quad (3)$$

Theorem

Let $\varphi(t, s, i)$ denote the unique solution in the class of functions belonging to $C(\overline{\mathcal{D}}) \bigcap C^{1,2}(\mathcal{D})$, with at most polynomial growth, of the problem (2, 3). Then

- $\varphi(t, S_t, X_t)$ *is the locally risk minimizing option price at* t.
- *An optimal strategy* $\pi^* = \{\pi^0_t, \pi^1_t\}$ *is given by*

$$\pi^1_t = \frac{\partial}{\partial s}\varphi(t, S_t, X_{t-}) \tag{4}$$

$$\pi^0_t = B_t^{-1}\varphi(t, S_t, X_{t-}) - \pi^1_t S^*_t \tag{5}$$

Proof.

Let $0 \leq t \leq T$. By applying Itô's formula to $e^{-\int_0^t r(X_u)du}\varphi_c(t, S_t, X_t)$ under the measure P and using (2), we obtain after suitable rearrangement of terms

$$
\begin{aligned}
& e^{-\int_0^t r(X_u)du}\varphi(t, S_t, X_t) \\
&= \varphi(0, S_0, X_0) + \int_0^t \frac{\partial \varphi(u, S_u, X_{u-})}{\partial s} dS_u^* + \int_0^t e^{-\int_0^u r(X_v)dv} \\
&\quad \int_{\mathbb{R}} [\varphi(u, S_u, X_{u-} + h(X_{u-}, z)) - \varphi(u, S_u, X_{u-})]\hat{\wp}(du, dz).
\end{aligned}
$$

$$
\begin{aligned}
& B_T^{-1}(S_T - K)^+ \\
&= \varphi(0, S_0, X_0) + \int_0^T \frac{\partial \varphi(u, S_u, X_{u-})}{\partial s} dS_u^* + \int_0^T e^{-\int_0^u r(X_v)dv} \\
&\quad \int_{\mathbb{R}} [\varphi(u, S_u, X_{u-} + h(X_{u-}, z)) - \varphi(u, S_u, X_{u-})]\hat{\wp}(du, dz).
\end{aligned}
$$

Hence $\frac{(S_T - K)^+}{B_T}$ has above *Föllmer-Schweizer decomposition.*

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$$
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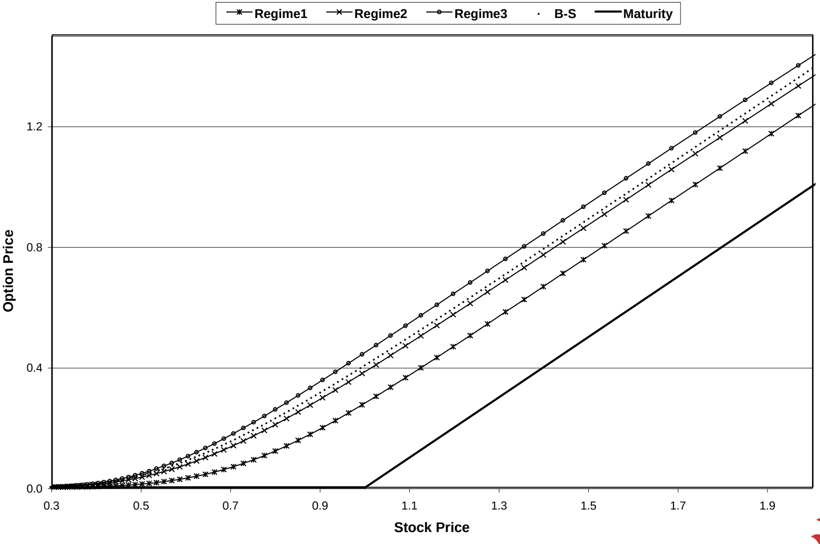
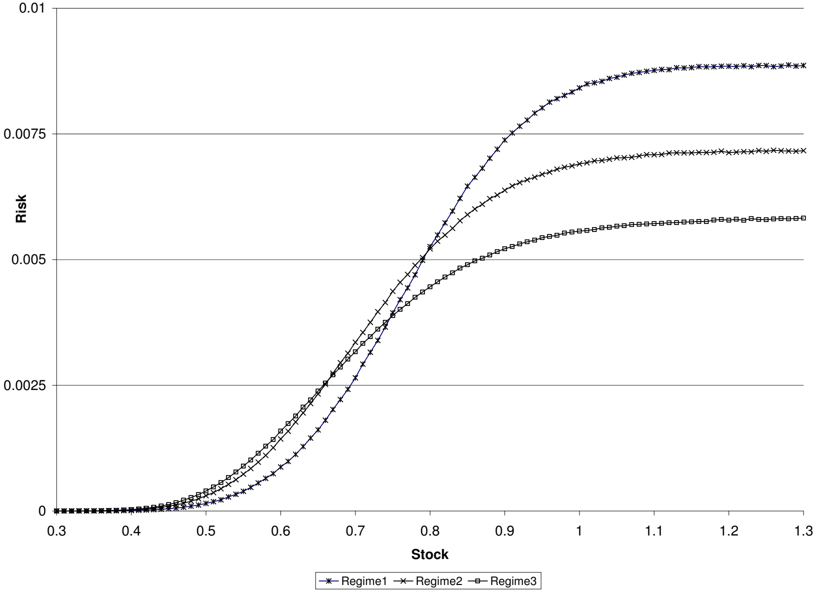


Figure : European Call Option


0.01
0.0075
0.005
0.0025
0
Risk
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
1.1
1.2
1.3
Stock
Regime1
Regime2
Regime3

IEs for Pricing & Hedging

Next we would see that price and hedging of call option satisfies systems of Volterra equations of second kind.

Theorem

The following integral equation has unique solution $\{\varphi(t,s,i),\quad i=1,2,\ldots,k\}$ *in the class of functions belonging to* $C(\overline{\mathcal{D}})\bigcap C^{1,2}(\mathcal{D})$

$$\varphi(t,s,i) = e^{-\lambda_i(T-t)}\eta_i(t,s) + \int_0^{T-t}\lambda_i \frac{e^{-(\lambda_i+r(i))v}}{\sigma(i)}$$

$$\times \sum_j p_{ij}\int_0^\infty \varphi(t+v,x,j)\frac{e^{-\frac{1}{2}\left(\left(\ln(\frac{x}{s})-(r(i)-\frac{\sigma^2(i)}{2})v\right)\frac{1}{\sigma(i)\sqrt{v}}\right)^2}}{\sqrt{2\pi}\sqrt{v}x}dxdv \qquad (6)$$

$$with\ \varphi(T,s,i) = (s-K)^+, \qquad \varphi(t,0,i)=0\ \forall t\in[0,T], i\in\chi \qquad (7)$$

where $\eta_i(t,s)$ *is the standard Black-Scholes price of European call option with fixed interest rate* $r(i)$ *and volatility* $\sigma(i)$.
Moreover, the solution $\varphi(t,s,i)$ *of* (6) *and* (7) *is the locally risk minimizing price of* $(S_T-K)^+$ *at time* t *with* $S_t=s, X_t=i$.

Theorem

Let ψ be given in the following manner

$$\psi(t,s,i) = e^{-\lambda_i(T-t)}\frac{\partial \eta_i(t,s)}{\partial s} + \int_0^{T-t} \lambda_i \frac{e^{-(\lambda_i+r(i))v}}{\sigma(i)^3} \sum_j p_{ij} \int_0^\infty \varphi(t+v,x,j)$$

$$\times \frac{e^{-\frac{1}{2}\left(\left(\ln(\frac{x}{s})-(r(i)-\frac{\sigma^2(i)}{2})v\right)\frac{1}{\sigma(i)\sqrt{v}}\right)^2}}{\sqrt{2\pi}v^{3/2}xs}\left(\ln\left(\frac{x}{s}\right)-\left(r(i)-\frac{\sigma^2(i)}{2}\right)v\right)dxdv$$

where $\varphi(t,s,i)$ is the solution of (6) and (7). Then $\xi_t := \psi(t,S_t,X_t)$ and $\varepsilon_t := \varphi^*(t,S_t,X_t) - \xi_t S_t^*$ comprise the optimal hedging strategy for the claim $(S_T-K)^+$.

Let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ be a complete probability space which holds a standard Brownian motion $\tilde{W}$ and a Markov process $\tilde{X}$ independent of $\tilde{W}$ such that the rate matrix of $\tilde{X}$ is same as X and $\tilde{\mathcal{F}}_t$ be the underlying filtration satisfying usual hypothesis. Let $\tilde{S}_t$ be given by

$$d\tilde{S}_t = \tilde{S}_t(r(\tilde{X}_t)dt + \sigma(\tilde{X}_t)d\tilde{W}_t), \ \tilde{S}_0 > 0.$$

Thus $\tilde{P}$ is risk-neutral measure for the risky asset $\tilde{S}$. Let Y_t represent the amount of time the process $\tilde{X}_t$ is at the current state after the last jump. Let the jump times be $0 = T_0 < T_1 < T_2 < \cdots$ and $n(t) = \max\{n \geq 0 \mid T_n \leq t\}$. Hence, $T_{n(t)} = t - Y_t$. Clearly, $f(y|i) := \lambda_i e^{-\lambda_i y}$ is the conditional probability density function of holding time and $F(y|i) = 1 - e^{-\lambda_i y}$ is the corresponding CDF where $\lambda_i = -\lambda_{ii}$. Here we recall the following obvious relation

$$\frac{f(y|i)}{1 - F(y|i)} = \lambda_i.$$

Using Markovity of $(\tilde{S}_t, \tilde{X}_t)$ we define

$$\varphi(t, \tilde{S}_t, \tilde{X}_t) := \tilde{E}[e^{-\int_t^T r(\tilde{X}_u)du}(\tilde{S}_T - K)^+ \mid \tilde{\mathcal{F}}_t] \quad (8)$$

where $\tilde{E}$ is expectation under $\tilde{P}$. Now we rewrite

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where $\tilde{E}$ is expectation under $\tilde{P}$. Now we rewrite

$$
\begin{aligned}
&\varphi(t,\tilde S_t,\tilde X_t)\\
&= \tilde E[e^{-\int_t^T r(\tilde X_u)du}(\tilde S_T-K)^+|\tilde S_t,\tilde X_t]\\
&= \tilde E[\tilde E[e^{-\int_t^T r(\tilde X_u)du}(\tilde S_T-K)^+|\tilde S_t,\tilde X_t,T_{n(t)+1}]|\tilde S_t,\tilde X_t]\\
&= \tilde P(T_{n(t)+1}>T|\tilde X_t)\tilde E[e^{-\int_t^T r(\tilde X_u)du}(\tilde S_T-K)^+|\tilde S_t,\tilde X_t,\{T_{n(t)+1}>T\}]+\\
&\int_0^{T-t}\tilde E[e^{-\int_t^T r(\tilde X_u)du}(\tilde S_T-K)^+|\tilde S_t,\tilde X_t,T_{n(t)+1}=t+v]\frac{f(t+v-T_{n(t)}\mid \tilde X_t)}{1-F(t-T_{n(t)}|\tilde X_t)}dv\\
&= e^{-\lambda_{\tilde x_t}(T-t)}\eta_{\tilde X_t}(t,\tilde S_t)+\int_0^{T-t}\lambda_{\tilde X_t}e^{-(\lambda_{\tilde X_t}+r(\tilde X_t))v}\sum_j p_{\tilde X_t j}\times\\
&\quad\int_0^\infty \tilde E[e^{-\int_{t+v}^T r(\tilde X_u)du}(\tilde S_T-K)^+|\tilde S_{t+v}=x,\tilde X_{t+v}=j,\tilde T_{n(t)+1}=t+v]\\
&\quad\frac{e^{-\frac12((\ln(\frac{x}{\tilde S_t})-(r(\tilde X_t)-\frac{\sigma^2(\tilde X_t)}{2})v)\frac{1}{\sigma(\tilde X_t)\sqrt v})^2}}{\sqrt{2\pi}\sigma(\tilde X_t)\sqrt{v}x}dxdv
\end{aligned}
$$

$$= e^{-\lambda_{\tilde{X}_t}(T-t)}\eta_{\tilde{X}_t}(t,\tilde{S}_t) + \int_0^{T-t} \lambda_{\tilde{X}_t} e^{-(\lambda_{\tilde{X}_t}+r(\tilde{X}_t))v} \times \sum_j p_{\tilde{X}_t j} \int_0^\infty \varphi(t+v,x,j) \frac{e^{-\frac{1}{2}\left((\ln(\frac{x}{\tilde{S}_t})-(r(\tilde{X}_t)-\frac{\sigma^2(\tilde{X}_t)}{2})v)\frac{1}{\sigma(\tilde{X}_t)\sqrt{v}}\right)^2}}{\sqrt{2\pi}\sigma(\tilde{X}_t)\sqrt{v}x} dxdv.$$

Thus φ solves the following equation

$$\varphi(t,s,i) = e^{-\lambda_i(T-t)}\eta_i(t,s) + \int_0^{T-t} \lambda_i e^{-(\lambda_i+r(i))v} \times \sum_j p_{ij} \int_0^\infty \varphi(t+v,x,j) \frac{e^{-\frac{1}{2}\left((\ln(\frac{x}{s})-(r(i)-\frac{\sigma^2(i)}{2})v)\frac{1}{\sigma(i)\sqrt{v}}\right)^2}}{\sqrt{2\pi}\sigma(i)\sqrt{v}x} dxdv$$

$$\varphi(T,s,i) = (s-K)^+.$$

The term $\varphi(t+v,x,j)$ is multiplied by $C^1((0,\infty))$ function in v and then integrated over $v \in (0, T-t)$. Hence, φ is in $C^{1,2}(\mathcal{D})$. RHS is image of φ under a contraction. Uniqueness follows.

$$
\begin{aligned}
&= e^{-\lambda_{\tilde{X}_t}(T-t)}\eta_{\tilde{X}_t}(t,\tilde{S}_t) + \int_0^{T-t} \lambda_{\tilde{X}_t} e^{-(\lambda_{\tilde{X}_t}+r(\tilde{X}_t))v} \\
&\quad \times \sum_j p_{\tilde{X}_t j} \int_0^\infty \varphi(t+v,x,j) \frac{e^{-\frac{1}{2}\left((\ln(\frac{x}{\tilde{S}_t})-(r(\tilde{X}_t)-\frac{\sigma^2(\tilde{X}_t)}{2})v)\frac{1}{\sigma(\tilde{X}_t)\sqrt{v}}\right)^2}}{\sqrt{2\pi}\sigma(\tilde{X}_t)\sqrt{v}x} dxdv.
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$$
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\varphi(t,s,i) &= e^{-\lambda_i(T-t)}\eta_i(t,s) + \int_0^{T-t} \lambda_i e^{-(\lambda_i+r(i))v} \\
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Note that $(\tilde{S}_t, \tilde{X}_t)$ is jointly Markov with infinitesimal generator $\tilde{\mathcal{A}}$ given by

$$\tilde{\mathcal{A}}\varphi(t,s,i) = \frac{1}{2}\sigma(i)^2 s^2 \frac{\partial^2 \varphi(t,s,i)}{\partial s^2} + r(i)s\frac{\partial \varphi(t,s,i)}{\partial s} + \sum_{j=1}^{k} \lambda_{ij}\varphi(t,s,i).$$

Therefore, (2) can be rewritten as

$$\frac{\partial \varphi}{\partial t}(t,s,i) + \tilde{\mathcal{A}}\varphi(t,s,i) = r(i)\varphi(t,s,i).$$

This implies that φ as in (8) is a mild solution of (2)-(3). It is also shown above that φ is in $C(\overline{\mathcal{D}}) \bigcap C^{1,2}(\mathcal{D})$. Hence φ is a classical solution of (2)-(3). Uniqueness is asserted from the stochastic representation. Hence, φ is the locally risk minimizing price of European call option at time t, $S_t = s, X_t = i$.

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Solving PDE

Replacing $t = T - v$ and $s = e^z$, get a new system of PDEs

$$-\frac{\partial\varphi(v,z,i)}{\partial v} + (r(i) - \frac{1}{2}\sigma(i)^2)\frac{\partial\varphi(v,z,i)}{\partial z} + \frac{1}{2}\sigma(i)^2\frac{\partial^2\varphi(v,z,i)}{\partial z^2} + \sum_{j=1}^{k}\lambda_{ij}\varphi(v,z,j) = r(i)\varphi(v,z,i)$$

on the domain $(0, T) \times \mathbb{R} \times \chi$ with $\varphi(0,z,i) = (e^z - K)^+$.

Let $N := [\frac{T}{\Delta t}]$, for $n \le N$, $m = 0, 1, \ldots M$,

$\varphi^n_m(i) := \varphi(n\Delta t, z_0 + m\Delta z, i)$, $\varphi^0_m(i) = (e^{z_0 + m\Delta z} - K)^+$.

Let $\varphi^n := [\varphi^n_0(1), \ldots, \varphi^n_0(k), \varphi^n_1(1), \ldots, \varphi^n_M(1), \ldots, \varphi^n_M(k)] \in \mathbb{R}^{k(M+1)}$.

Now the Crank Nicholson discretization gives

$$A\varphi^{n+1} = (-2I - A)\varphi^n$$

where A is an appropriate block diagonal matrix of order $k(M+1)$. By repeated use, the numerical solution to the PDE

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Let $N := [\frac{T}{\Delta t}]$, for $n \leq N$, $m = 0, 1, \ldots M$,
$\varphi_m^n(i) := \varphi(n\Delta t, z_0 + m\Delta z, i)$, $\varphi_m^0(i) = (e^{z_0+m\Delta z} - K)^+$.
Let $\varphi^n := [\varphi_0^n(1), \ldots, \varphi_0^n(k), \varphi_1^n(1), \ldots, \varphi_M^n(1), \ldots, \varphi_M^n(k)] \in \mathbb{R}^{k(M+1)}$.
Now the Crank Nicholson discretization gives

$$A\varphi^{n+1} = (-2I - A)\varphi^n$$

where A is an appropriate block diagonal matrix of order $k(M+1)$. By repeated use, the numerical solution to the PDE

$$\varphi^n = (-2A^{-1} - I)^n\varphi^0.$$

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Theorem

The C-N scheme given above is stable for

$$\Delta z < \min_i \left| \frac{2\sigma^2(i)}{2r(i) - \sigma^2(i)} \right|.$$

Proof. Let λ_A be an eigen value of A. Using Gerschgorin's circle theorem (G. D. Smith p. 88) for sufficiently small Δz, i.e. $\Delta z < \min_i \left| \frac{2\sigma^2(i)}{2r(i) - \sigma^2(i)} \right|$, we have $\Re(\lambda_A) \leq -1 - \frac{\Delta t}{2} r(i)$ where $\Re(\lambda_A)$ is the real part of λ_A. Thus the real part of all the eigen values of A are less than -1, which implies

$$\left| \frac{2}{\lambda_A} + 1 \right| < 1.$$

Hence the modulus of all the eigen values of $(-2A^{-1} - I)$ is less than 1. This gives the stability of the iterative scheme. □

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Solving Volterra IE

The equation is solved using a step-by-step quadrature method.

$$\mathcal{G}(m, m', l, i) := \frac{e^{-\frac{1}{2}((\ln(\frac{m'}{m}) - (r(i) - \frac{\sigma^2(i)}{2}) l\Delta t) \frac{1}{\sigma(i)\sqrt{l\Delta t}})^2}}{\sqrt{2\pi}\sigma(i) m' \Delta s \sqrt{l\Delta t}}$$

$$\varphi_m^n(i) \approx \varphi(T - n\Delta t, m\Delta s, i).$$

$$\begin{aligned} \varphi_m^n(i) &= e^{-\lambda_i n\Delta t}\eta_i(T - n\Delta t, m\Delta s) \\ &\quad + \Delta t \sum_{l=1}^{n} \omega_n(l) e^{-l\Delta t(r(i)+\lambda_i)} \sum_j p_{ij}\Delta s \sum_{m'} \varphi_m^{n-1}(j)\mathcal{G}(m, m', l, i) \\ \varphi_m^0(i) &= (m\Delta s - K)^+. \end{aligned}$$

We choose a repeated trapezium rule, i.e., the weights ω_n are

$$\omega_n(l) = \begin{cases} 1, & \text{for } l = 1, 2, \ldots, n-1 \\ \frac{1}{2}, & \text{for } l = 0, n. \end{cases}$$

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Stability Result

Theorem

Let $a := \max_{\chi} \lambda_i e^{-(\lambda_i + r(i))}$. *For*

$$\Delta t \leq \frac{e^{-aT}}{a} \tag{9}$$

the scheme (17) is strictly stable with respect to an isolated perturbation. Moreover, the scheme displays uniformly bounded error propagation.

$$\varepsilon_n = a\Delta t(1 + a\Delta t)^{N-n}\delta, \quad \varepsilon := \sum_{n=0}^{N-1} \varepsilon_n \; < \; (e^{aT} - 1)\delta.$$

The accumulated effect (ε) of perturbations δ, added at each step in $\varphi_m^n(i)$ is uniformly bounded by a constant multiple of δ.

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Theorem

Given a finite grid of the domain $[0, T] \times \overline{\mathbb{R}_+}$, let N and M be the number of discrete points on $[0, T]$ and $\overline{\mathbb{R}_+}$ respectively. Let $T(N, M)$ denotes the computational complexity to solve (6) and (7) with above grid using step by step quadrature method. Then we have

$$T(N, M) = O(N^2 M^2).$$

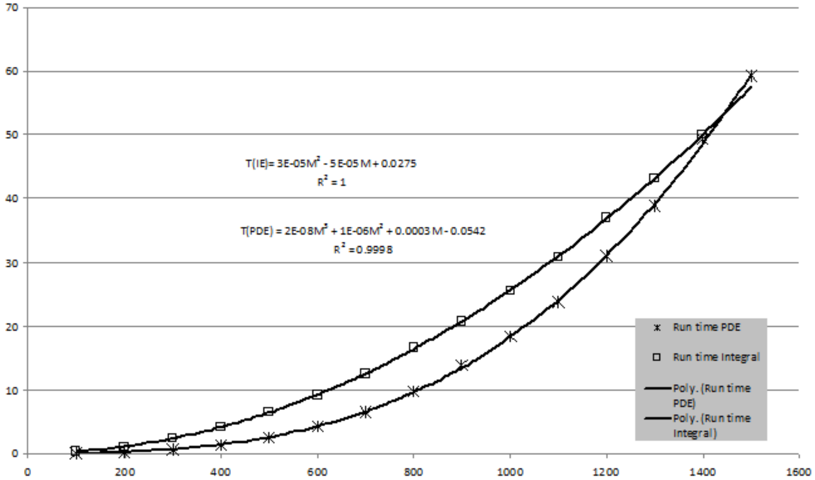

Figure : Run time ($\approx T(N, M)$) are plotted for both of solving PDE and IE

Thank You