Risk Minimizing Option Pricing in a Semi-Markov Modulated Market [1]

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Overview

1. Incompleteness of the Market Model
   - Model Description
   - Pricing in an Incomplete Market
   - Differential Equation of Price Function
2. Numerical Methods
   - Quadrature Method and Stability
   - Numerical Example

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Semi-Markov Modulated Market

Keeping the drawbacks of Black-Scholes-Merton model in mind, we model the floating interest rate r and volatility σ as semi-Markov processes (non-constant w.r.t time).

- $\{X_t\}_{t\geq 0}$ be a semi-Markov process taking values in $\mathcal{X} = \{1, 2, \ldots, k\}$.
- X_t is modeled to present hypothetical states of the market at time t.
- Let $r(X_t)$ and $\sigma(X_t)$ be the floating interest rate and market volatility at time t.

This market has no arbitrage but is incomplete. Thus

- An arbitrage free option price is not unique.
- Every contingent claim may not be attainable by a self-financing strategy.

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Model Description

Let T_n be the time instance of nth jump, $i, j \in \mathcal{X}$, $t > 0$

$$P(X_{T_{n+1}} = j, T_{n+1} - T_n \le t \mid X_{T_n} = i) = p_{ij} F(t \mid i).$$

- The transition matrix (p_{ij}) is irreducible.
- $F(t \mid i) < 1$ for each i and for all $t \in [0, \infty)$.
- $F(\cdot \mid i)$ has continuously differentiable density $f(\cdot \mid i)$.

$$\begin{aligned} B_t &= e^{\int_0^t r(X_u) du} \\ dS_t &= S_t \left(\mu(X_t) dt + \sigma(X_t) dW_t\right), \qquad S_0 > 0. \end{aligned}$$

$\{X_t\}$ and $\{W_t\}$ are independent.

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Embed $\mathcal{X}$ in $\mathbb{R}^k$ by identifying i with $e_i \in \mathbb{R}^k$. For $y \in [0, \infty)$, $i, j \in \mathcal{X}$ let

$$\lambda_{ij}(y) := p_{ij} \frac{f(y \mid i)}{1 - F(y \mid i)} \geq 0 \text{ for } i \neq j, \; \lambda_{ii}(y) := - \sum_{j \in \mathcal{X}, j \neq i} \lambda_{ij}(y) \text{ for } i \in$$

For $i \neq j \in \mathcal{X}, y \in \mathbb{R}_+$, let $\Lambda_{ij}(y)$ be consecutive (with respect to the lexicographic ordering on $\mathcal{X} \times \mathcal{X}$) left closed right open intervals of the positive line, each having length $\lambda_{ij}(y)$. Define the functions $h : \mathcal{X} \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}^k$ and $g : \mathcal{X} \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ by

$$h(i, y, z) := \begin{cases} j - i & \text{if } z \in \Lambda_{ij}(y) \\ 0 & \text{otherwise,} \end{cases}$$

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Consider the process $\{X'_t, Y_t\}$ described by the following stochastic integral equations

$$\left.\begin{aligned} X'_t &= X'_0 + \int_0^t \int_{\mathbb{R}} h(X'_{u-}, Y_{u-}, z)\wp(du, dz) \\ Y_t &= t - \int_0^t \int_{\mathbb{R}} g(X'_{u-}, Y_{u-}, z)\wp(du, dz) \end{aligned}\right\} \quad (1)$$

where the integrations are over the interval $(0, t]$ and $\wp(dt, dz)$ is an $\mathcal{M}(\mathbb{R}_+ \times \mathbb{R})$ -valued Poisson random measure with intensity $dtdz$, independent of X'_0, a $\mathcal{X}$-valued random variable.

Theorem

The process $\{X'_t\}$ defined in (1) is a semi-Markov process with transition probability matrix (p_{ij}) and conditional holding time distributions $F(y \mid i)$.

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Proof.

From (1) it is clearly seen that X'_t is a right continuous (since the integrations are over $(0, t]$) jump process taking values in $\mathcal{X}$. For a fixed $\omega \in \Omega$, $\{X'_t(\omega)\}$ has a jump at t_0 to a state j iff $\wp\left(\{t_0\} \times \Lambda_{X'_{t_0-}(\omega)j}(Y_{t_0-}(\omega))\right)(\omega) \neq 0$. Using this inductively for each jump we see that, $Y_{t_0}(\omega) = 0$ if and only if $\{X'_t(\omega)\}$ has a jump at t_0.

T_n = Time of nth jump of X'_t & $T_0 := 0$

$\tau_n := T_n - T_{n-1}$ & $n(t) := \max\{n : T_n \leq t\}$. Thus

$T_{n(t)} \leq t < T_{n(t)+1}$ & $Y_t = t - T_{n(t)}$.

Using the property of Poison random measure

$$P\left(\text{No jump in } (T_n, T_n + y] \mid X'_{T_n} = i\right) = \exp\left(-\int_0^y \sum_{j \neq i} \lambda_{ij}(s) ds\right).$$

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$$\frac{d}{dy}F(y \mid i) = (1 - F(y \mid i))\sum_{j\neq i}\lambda_{ij}(y)$$

which gives $\exp\left(-\int_0^y \sum_{j\neq i}\lambda_{ij}(s)ds\right) = 1 - F(y \mid i)$. Thus

$$P(\tau_{n+1} \leq y \mid X'_{T_n} = i) = F(y \mid i).$$

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$$\begin{aligned} & P(X'_{T_{n+1}} = j, \tau_{n+1} \leq y \mid X'_{T_n} = i) \\ = & \int_0^y \exp\left(-\int_0^u \sum_{j\neq i}\lambda_{ij}(s)ds\right)\lambda_{ij}(u)du \\ = & \int_0^y \Big(1 - F(u \mid i)\Big) p_{ij}\frac{f(u \mid i)}{1 - F(u \mid i)}du \\ = & \; p_{ij}F(y \mid i). \end{aligned}$$

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The equations together show $\{X'_t\}$ is a semi-Markov process with transition probability matrix (p_{ij}) and conditional holding time distributions $F(y \mid i)$. □

In view of Theorem we write $X'_t = X_t$ from now on.

We make the following assumption

(A2) $\wp(dt, dz), X_0, W$ and S_0, defined on $(\Omega, \mathcal{F}, P)$ are independent.

It is clear that the process (S_t, X_t, Y_t), defined on $(\Omega, \mathcal{F}, P)$ is jointly Markov. We now compute the infinitesimal generator of the Markov process (S_t, X_t, Y_t).

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Let $\varphi : \mathbb{R} \times \mathcal{X} \times \mathbb{R}_+ \to \mathbb{R}$ be a smooth function. Then by Itô's formula

$$\begin{aligned}
& d\varphi(S_t, X_t, Y_t) \\
= & \int_{\mathbb{R}} \Big\{ \varphi\Big(S_t, X_{t-} + h(X_{t-}, Y_{t-}, z), Y_{t-} - g(X_{t-}, Y_{t-}, z)\Big) \\
& \qquad - \varphi(S_t, X_{t-}, Y_{t-}) \Big\} \wp(dt, dz) + \frac{\partial}{\partial s}\varphi(S_t, X_{t-}, Y_{t-}) dS_t \\
& + \frac{1}{2}\frac{\partial^2}{\partial s^2}\varphi(S_t, X_{t-}, Y_{t-}) d\langle S, S\rangle_t + \frac{\partial}{\partial y}\varphi(S_t, X_{t-}, Y_{t-}) dt \\
= & \mu(X_{t-}) S_t \frac{\partial}{\partial s}\varphi(S_t, X_{t-}, Y_{t-}) dt \\
& + \frac{1}{2}\sigma^2(X_{t-}) S_t^2 \frac{\partial^2}{\partial s^2}\varphi(S_t, X_{t-}, Y_{t-}) dt + \frac{\partial}{\partial y}\varphi(S_t, X_{t-}, Y_{t-}) dt \\
& + \sum_{j \neq X_{t-}} [\varphi(S_t, j, 0) - \varphi(S_t, X_{t-}, Y_{t-})] \lambda_{X_{t-}, j}(Y_{t-}) dt + d\tilde{M}_t
\end{aligned}$$

where $\tilde{M}_t$ is a martingale given by

$$\tilde{M}_t = \tilde{M}_0 + \int_0^t S_u\sigma(X_{u-})\frac{\partial}{\partial s}\varphi(S_u, X_{u-}, Y_{u-})dW_u + \int_0^t\int_{\mathbb{R}}\Big\{\varphi\Big(S_u, X_{u-} + h(X_{u-}, Y_{u-}, z), Y_{u-} - g(X_{u-}, Y_{u-}, z)\Big) - \varphi(S_u, X_{u-}, Y_{u-})\Big\}\hat{\wp}(du, dz)$$

where $\hat{\wp}(dt, dz) := \wp(dt, dz) - dtdz$ is the compensated Poisson random measure. If $\mathcal{A}$ denotes the infinitesimal generator of (S_t, X_t, Y_t), then we obtain

$$\mathcal{A}\varphi(s, i, y) = \frac{\partial}{\partial y}\varphi(s, i, y) + \mu(i)s\frac{\partial}{\partial s}\varphi(s, i, y) + \frac{1}{2}\sigma^2(i)s^2\frac{\partial^2}{\partial s^2}\varphi(s, i, y) + \frac{f(y \mid i)}{1 - F(y \mid i)}\sum_{j\neq i} p_{ij}[\varphi(s, j, 0) - \varphi(s, i, y)].$$

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Incompleteness of Market

REMARK

The semi-Markov modulated market is not complete, the claim H may not be attained through a self financing strategy. An extra cash flow is required to replicate H at the maturity time. Therefore, to price an option the writer must take care of the risk associated with a hedging strategy. Hence he would like to seek a "optimal hedging strategy", causing 'minimum' cash flow and a "risk minimizing" price corresponding to that.

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Minimal Martingale Measure

For $i \in \mathcal{X}$, let $\gamma(i) := \frac{\mu(i)-r(i)}{\sigma(i)}$, i.e., $\gamma(i)$ is the **market price of risk** in regime (or mode) i. Let

$$\begin{aligned} \rho_t &:= \exp\Big(- \int_0^t \gamma(X_u) dW_u - \frac{1}{2}\int_0^t \gamma(X_u)^2 du \Big), \qquad t \leq T \\ dP^* &:= \rho_T \; dP \end{aligned}$$

Note that P^*, as defined above is a probability measure and is equivalent to P. The P^* has some special properties, viz., it is the minimal martingale measure.

Definition

An EMM $P' \equiv P$ is said to be **minimal** if $P' = P$ on $\mathcal{F}_0$ and if any square integrable P- martingale which is orthogonal to martingale part of S_t under P remains a martingale under P'.

Additional Cash Flow and Local Risk Minimization

For a given strategy π let us define

$$\begin{aligned} V_t(\pi) &= \pi_t^0 B_t + \pi_t^1 S_t \\ V_t^*(\pi) &= \frac{V_t(\pi)}{B_t} \\ C_t^*(\pi) &:= V_t^*(\pi) - \int_0^t \pi_u^1 dS_u^*, \qquad 0 \leq t \leq T, \\ R_t(\pi) &:= E[(C_T^*(\pi) - C_t^*(\pi))^2 \mid \mathcal{F}_t]. \end{aligned}$$

Definition

An admissible strategy π^* is said to be **optimal** (i.e., locally risk minimizing) if the corresponding discounted cost $\{C_t^*(\pi^*)\}$ is a square integrable martingale orthogonal to

$$M_t := \int_0^t \sigma(X_u) S_u^* dW_u.$$

Let $H := (S_T - K)^+$, $H^* = B_T^{-1} H$.

The existence of an optimal strategy $\equiv$ The existence of **Föllmer-Schweizer decomposition** of H^* in the form

$$H^* = H_0 + \int_0^T \xi_u^{H^*} dS_u^* + L_T^{H^*} \qquad \text{under } P. \tag{2}$$

$H_0 \in L^2(\Omega, \mathcal{F}_0, \mathcal{P})$,
$\xi^{H^*} = \{\xi_t^{H^*}\}$ satisfies the integrability condition,
and $L^{H^*} = \{L_t^{H^*}, 0 \le t \le T\}$ is a zero-mean square integrable martingale orthogonal to $\{M_t,\ 0 \le t \le T\}$.

For the decomposition (2), the associated optimal strategy $\pi = (\pi_t^0, \pi_t^1)$ is given by

$$\pi_t^1 := \xi_t^{H^*}, \quad \pi_t^0 := V_t^* - \xi_t S_t^*. \tag{3}$$

with

$$V_t^* := H_0 + \int_0^t \xi_u^{H^*} dS_u^* + L_t^{H^*}, \qquad 0 \le t \le T. \tag{4}$$

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$$V_t^* := H_0 + \int_0^t \xi_u^{H^*} dS_u^* + L_t^{H^*}, \qquad 0 \leq t \leq T. \tag{4}$$

Thus the discounted optimal cost $C_t^*(\pi)$ is given by

$$C_t^*(\pi) = H_0 + L_t^{H^*}.$$

The locally risk minimizing price of call option is given by

$$\begin{aligned} & B_t V_t^* \\ &= E^*[B_t H^* \mid \mathcal{F}_t] \\ &= E^*[e^{-\int_t^T r(X_u)du}(S_T - K)^+ \mid \mathcal{F}_t] \\ &= E^*[e^{-\int_t^T r(X_u)du}(S_T - K)^+ \mid S_t, X_t, Y_t] \end{aligned}$$

where, $\mathcal{F}_t = \sigma(X_u, S_u, u \leq t)$ and Y_t is the time spent at the current state X_t after the last jump.

Call Option in Semi-Markov Modulated Market

Consider the following equation (5) which is the generalization of Black-Scholes-Merton partial differential equation for the semi-Markov modulated market.

$$\frac{\partial}{\partial t}\varphi(t,s,i,y) + \frac{\partial}{\partial y}\varphi(t,s,i,y) + r(i)s\frac{\partial}{\partial s}\varphi(t,s,i,y) + \frac{1}{2}\sigma^2(i)s^2\frac{\partial^2}{\partial s^2}\varphi(t,s,i,y) + \frac{f(y \mid i)}{1-F(y \mid i)}\sum_{j\neq i} p_{ij}[\varphi(t,s,j,0) - \varphi(t,s,i,y)] = r(i)\,\varphi(t,s,i,y), \quad (5)$$

defined on

$$\mathcal{D} := \{(t,s,i,y) \in (0,T) \times \mathbb{R} \times \mathcal{X} \times (0,T) \mid y \in (0,t)\} \quad (6)$$

with the terminal condition

$$\varphi(T,s,i,y) = (s-K)^+; \quad s \in \mathbb{R}; \quad 0 \le y \le T; \quad i = 1,2,\ldots,k. \quad (7)$$

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Theorem

(i) The Cauchy problem (5, 7) has a unique solution in the class of functions belonging to $C(\overline{\mathcal{D}}) \bigcap C^{1,2,1}(\mathcal{D})$*, with at most polynomial growth where* $\mathcal{D}$ *is as in (6).*
(ii) Let $\varphi(t, s, i, y)$ *denote this unique solution. Then*

$$\varphi(t, S_t, X_t, Y_t)$$

is the locally risk minimizing price of call option at time t*.*
(iii) An optimal strategy $\pi^* = \{\xi_t^*, \eta_t^*\}$ *is given by*

$$\begin{aligned} \xi_t^* &= \frac{\partial}{\partial s}\varphi(t, S_t, X_{t-}, Y_{t-}) \\ \eta_t^* &= B_t^{-1}\varphi(t, S_t, X_{t-}, Y_{t-}) - \xi_t^* S_t^*. \end{aligned}$$

Proof

Let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ be a probability space, $\tilde{W}$ a standard Brownian motion on it and $\tilde{X}$ a semi-Markov process independent of $\tilde{W}$. The distribution of $\tilde{X}$ is as same as X. Let $\tilde{S}_t$ be given by

$$d\tilde{S}_t = \tilde{S}_t(r(\tilde{X}_t)dt + \sigma(\tilde{X}_t)d\tilde{W}_t), \tilde{S}_0 > 0.$$

Then $(\tilde{S}_t, \tilde{X}_t, \tilde{Y}_t)$ is jointly Markov with generator $\tilde{\mathcal{A}}$ given by

$$\tilde{\mathcal{A}}\varphi(s,i,y) = \frac{\partial}{\partial y}\varphi(s,i,y) + r(i)s\frac{\partial}{\partial s}\varphi(s,i,y) + \frac{1}{2}\sigma^2(i)s^2 \times \frac{\partial^2}{\partial s^2}\varphi(s,i,y) + \frac{f(y \mid i)}{1 - F(y \mid i)}\sum_{j \neq i} p_{ij}[\varphi(s,j,0) - \varphi(s,i,y)].$$

Let

$$\varphi(t, \tilde{S}_t, \tilde{X}_t, \tilde{Y}_t) := \tilde{E}[e^{-\int_t^T r(\tilde{X}_u)du}(\tilde{S}_T - K)^+ \mid \tilde{S}_t, \tilde{X}_t, \tilde{Y}_t].$$

Then $\varphi(t,s,i,y)$ is a mild solution of (5) with condition (7).

Proof (Continued)

To establish the desired regularity we derive

$$\varphi(t,s,i,y)$$
$$= \frac{1-F(T-t+y \mid i)}{1-F(y \mid i)}\eta_i(t,s) + \int_0^{T-t} \frac{e^{-r(i)v}}{\sigma(i)\sqrt{v}} \frac{f(y+v \mid i)}{1-F(y \mid i)} \times$$
$$\sum_j p_{ij} \int_0^\infty \varphi(t+v,x,j,0) \frac{e^{\frac{-1}{2}((\ln(\frac{x}{s})-(r(i)-\frac{\sigma^2(i)}{2})v)\frac{1}{\sigma(i)\sqrt{v}})^2}}{\sqrt{2\pi}x} dxdv$$

where $\eta_i(t,s)$ is the standard Black-Scholes price of call option with fixed interest rate and volatility $r(i)$ and $\sigma(i)$ respectively. Under the model assumption the second term is in $C^{1,2,1}(\mathcal{D})$. Hence $\varphi(t,s,i,y)$ is in $C^{1,2,1}(\mathcal{D})$.

Numerical Solution of the Non-local System

- For the semi-Markov modulated market the corresponding B-S-M equations are systems of (non-local) integro-partial differential equations on some non-rectangular domain.
- A finite difference scheme is computationally expensive due to the extra dimensionality and non-rectangular nature of the domain.
- In order to overcome this difficulty we develop an alternative numerical scheme based on the discretization of the above integral equation. The equation is solved using a step-by-step quadrature method.

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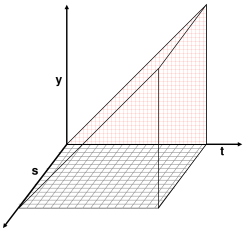

$$\mathcal{G}(m, m', l, i) := \frac{e^{\frac{-1}{2}\left(\left(\ln\left(\frac{m'}{m}\right) - \left(r(i) - \frac{\sigma^2(i)}{2}\right) l\Delta t\right)\frac{1}{\sigma(i)\sqrt{l\Delta t}}\right)^2}}{\sqrt{2\pi}\sigma(i) m' \Delta s \sqrt{l\Delta t}},$$

$$\varphi_m^n(i) \approx \varphi(T - n\Delta t, m\Delta s, i, 0)$$

$$\varphi_m^n(i) = (1 - F(n\Delta t \mid i))\eta_i(T - n\Delta t, m\Delta s) + \Delta t \sum_{l=1}^{n} \omega_n(l) e^{-r(i) l\Delta t} f(l\Delta t \mid i) \sum_j p_{ij} \Delta s \sum_{m'} \varphi_{m'}^{n-l}(j) \mathcal{G}(m, m', l, i)$$

Stability Result

Theorem

Let $a := \max_{\mathcal{X}\times[0,T]} e^{-r(i)v} f(v \mid i)$. *Then for*

$$\Delta t \le \frac{e^{-aT}}{a} \tag{8}$$

the scheme is strictly stable with respect to an isolated perturbation. Moreover the scheme displays uniformly bounded error propagation.

$$\varepsilon_n = a\Delta t(1 + a\Delta t)^{N-n}\delta, \quad \varepsilon := \sum_{n=0}^{N-1} \varepsilon_n \; < \; (e^{aT} - 1)\delta.$$

The accumulated effect (ε) of perturbations δ, added at each step in $\varphi_m^n(i)$ is uniformly bounded by a constant multiple of δ.

A Specific Semi-Markov Modulated Market

The state space $\mathcal{X} = \{1, 2, 3\}$.

$$\big(\mu(i), \sigma(i), r(i)\big) := \begin{cases} (0.2, 0.2, 0.2) & \text{if} \quad i = 1 \\ (0.6, 0.4, 0.5) & \text{if} \quad i = 2 \\ (0.8, 0.3, 0.7) & \text{if} \quad i = 3 \end{cases}.$$

The transition probability matrix is assumed to be given by

$$(p_{ij}) = \begin{pmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/3 & 2/3 & 0 \end{pmatrix}.$$

The holding time in each regime is assumed to be $\Gamma(2, 1)$. That is

$$f(y \mid i) = ye^{-y}, \ \ y \geq 0 \text{ and } i = 1, 2, 3.$$

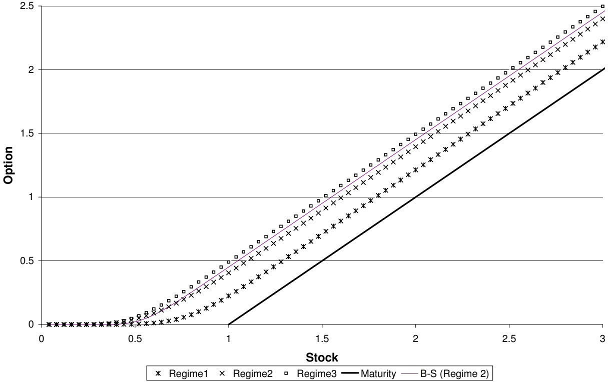

Figure : The risk minimizing price of European call option ($K = 1, T = 1$) for three different initial regimes

In a Specific Semi-Markov Modulated Market

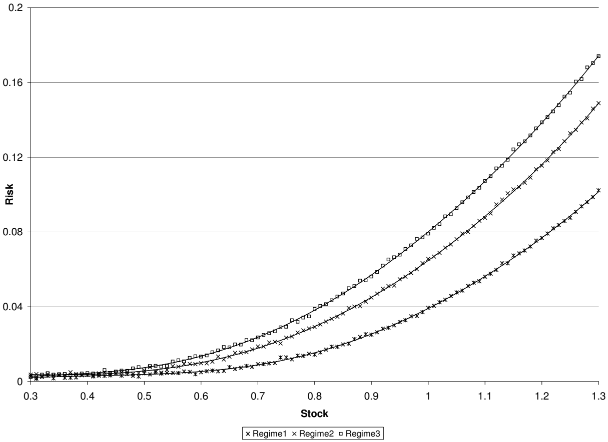

Figure : Quadratic Risks of Hedging European Call at Different Initial Conditions

In a Specific Semi-Markov Modulated Market

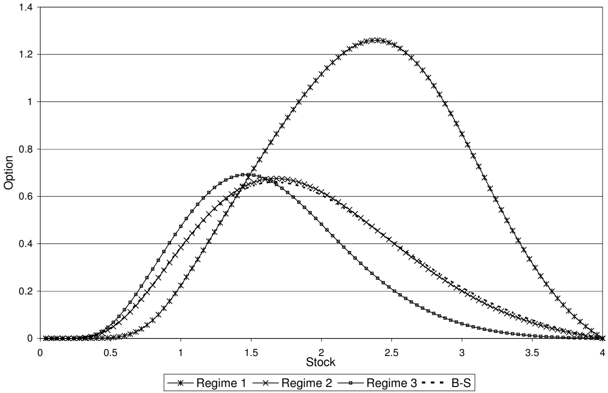

Figure : Price of Up-Out Barrier Option

Thank You