Risk Sensitive Portfolio Optimization in a Semi-Markov Modulated Market [1]

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"all the eggs should not be placed in the same basket"

- A financial market consists of numerous securities.
- A typical investor invests his/her initial capital in different securities and carry out continuous trading of them to increase the final wealth of the portfolio.

Q1. Which choice of portfolio would result in the best return?

Q2. Assets are risky. How to manage the risk?

Notations

- S_t^0:= locally risk free asset price at time t
- S_t^i:= risky asset prices at time t, $i = 1, \ldots, n$
- $N^i(t)$:= number of units invested in the i^{th} asset at t
- $V_t := \sum_{i=0}^n N^i(t) S_t^i$, value of the portfolio at t
- $u^i(t) := \frac{N^i(t) S_t^i}{V_t}$, the fraction invested in the i^{th} asset.

Then

$$\sum_{i=0}^{n} u^i(t) = 1 .$$

Hence

$$u^0(t) = 1 - \sum_{i=1}^{n} u^i(t).$$

SDE Satisfied by the Value of Portfolio

Let $u(t) = [u^1(t), \ldots, u^n(t)]'$ $(\in A \subseteq \mathbb{R}^n)$ be the portfolio at time t. Hence $u(t)$ is an A-valued process.
Self-financing condition implies that

$$dV_t = \sum_{i=0}^{n} N^i(t) dS^i_t.$$

Then the value of the portfolio or the wealth process denoted by V^u_t takes the form

$$\frac{dV^u_t}{V^u_t} = \sum_{i=0}^{n} u^i(t) \frac{dS^i_t}{S^i_t}.$$

Semi-Markov Process

- $\{X_t\}_{t\geq 0}$ be a semi-Markov process taking values in $\mathcal{X} = \{1, 2, \ldots, k\}$.
- X_t is modeled to present hypothetical states of the market at time t.

$$P(X_{T_{n+1}} = x', T_{n+1} - T_n \leq y \mid X_{T_n} = x) = p_{xx'} F(y \mid x).$$

- The transition matrix $(p_{xx'})$ is irreducible.
- $F(y \mid x) < 1$ for each x and for all $y \in [0, \infty)$.
- $F(\cdot \mid x)$ has continuously differentiable density $f(\cdot \mid x)$.

Define $Y_t := t - T_{n(t)}$ [Holding Time]

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Semi-Markov Modulated Market Model

$$
\begin{aligned}
dS_t^0 &= r(t,X_t)S_t^0 dt, \; S_0^0 = s_0 > 0\,, \\
dS_t^i &= S_t^i \left[\mu^i(t,X_t)dt + \sum_{j=1}^{n} \sigma_{ij}(t,X_t)\, dW_t^j \right], \; S_0^i = s_i > 0, \\
\sigma(t,x) &= [\sigma_{ij}(t,x)]_{n\times n}, \; x = 1,\cdots,k, \\
b(t,x) &= [\mu^1(t,x) - r(t,x),\cdots,\mu^n(t,x) - r(t,x)].
\end{aligned}
$$

The wealth V^u corresponding to the portfolio u satisfies

$$
dV_t^u = V_t^u[(r(t,X_t) + b(t,X_t)u(t))\, dt + u(t)'\sigma(t,X_t)dW_t]. \quad (1)
$$

(i) $r(t,x), \mu^i(t,x), \sigma_{ij}(t,x)$ are continuous on $[0,T]$ $\forall x, i, j$.
(ii) $a(t,x) := \sigma(t,x)\sigma(t,x)'$ continuous on $[0,T]$
and $\exists$ a $\delta > 0$ such that $a(t,x) \geq \delta I$ $\forall x$
(iii)$\{X_t\}$ and $\{W_t\}$ are independent.

Derivation - Infinitesimal Generator

For a fixed $u \in A$ the process $\{(V_t^u, X_t, Y_t)\}_{t\geq 0}$ is Markov. The infinitesimal generator of the process is the family of operators $\mathcal{A}_t^u : C^{2,1}(\mathbb{R}\times\mathcal{X}\times(0,t))\bigcap C(\mathbb{R}\times\mathcal{X}\times[0,t]) \to C(\mathbb{R}\times\mathcal{X}\times[0,t])$, given by

$$
\begin{aligned}
\mathcal{A}_t^u\varphi(v,x,y) := \frac{\partial}{\partial y}\varphi(v,x,y) + (r(t,x)+b(t,x)u)v\frac{\partial}{\partial v}\varphi(v,x,y) \\
+\frac{1}{2}(u'a(t,x)u)v^2\frac{\partial^2}{\partial v^2}\varphi(v,x,y) \\
+\frac{f(y\mid x)}{1-F(y\mid x)}\sum_{j\neq x} p_{xj}[\varphi(v,j,0)-\varphi(v,x,y)].
\end{aligned}
$$

$\varphi \in Dom(\mathcal{A}_t^u), v\in\mathbb{R}, x\in\mathcal{X}, y\in(0,t)$.

Expected Terminal Utility - Optimization - HJB

- Let U be a utility function satisfying the usual condition.
- The trader's objective: maximize expected terminal utility

$$\begin{aligned} J^u(t,v,x,y) &:= E^u[U(V_T^u) \mid V_t = v, X_t = x, Y_t = y] \\ \varphi(t,v,x,y) &:= \sup_u J^u(t,v,x,y) \end{aligned}$$

Supremum is taken over all admissible portfolio strategies.

- Using DPP and the verification theorem of controlled Markov processes, the HJB equation for φ is given by

$$\frac{\partial}{\partial t}\varphi(t,v,x,y) + \sup_{u\in A} \mathcal{A}_t^u \varphi(t,v,x,y) = 0, \quad \varphi(T,v,x,y) = U(v).$$

Can we solve it?

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Examples: Power Utility and Logarithmic Utility

If $U(v) = \log v, \ v > 0$ or $U(v) = \frac{1}{\gamma} v^{\gamma}, \ 0 < \gamma < 1$ then the corresponding HJB equation has the classical solution.

Sketch of the Proof

1. Using an appropriate trial solution we separate the variable v. Obtain a non-local differential equation involving variables t, x, y.
2. Using Feynman-Kac formula, we obtain a mild solution of the new equation.
3. Using stochastic analysis the mild solution is shown to satisfy a Volterra equation of second kind.
4. The desired smoothness of the solution is obtained by studying the Volterra equation.

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Risk Sensitive Criterion: Definitions & Motivation

For a particular $\theta(\neq 0)$ (to be set by the investor according to his risk aversion) the risk sensitive criterion on finite and infinite time horizon are defined as

$$J_\theta^{u,T}(t,v,x,y) := -\frac{2}{\theta}\log\, E^u\left[{V_T^u}^{\frac{-\theta}{2}} \mid V_t = v, X_t = x, Y_t = y\right]$$

and

$$J_\theta^{u}(t,v,x,y) := -\frac{2}{\theta}\liminf_{T\to\infty}\frac{1}{T}\,\log\, E^u\left[{V_T^u}^{\frac{-\theta}{2}} \mid V_t = v, X_t = x, Y_t = y\right].$$

Taylor series expansion of J_θ^T about $\theta = 0$ gives

$$-\frac{2}{\theta}\log\, E^u\left[e^{(-\frac{\theta}{2})\,\log V_T^u}\right] = E^u \log V_T^u - \frac{\theta}{4}\mathrm{Var}^u(\log V_T^u) + o(\theta).$$

A risk neutral (i.e., $\theta = 0$) investor would effectively optimize the expected logarithmic utility of terminal wealth.

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Finite Horizon Case

Theorem

(i) Let $\psi_\theta(t,x,y) = E[e^{\int_t^T h_\theta(s,X_s)ds} \mid X_t = x, Y_t = y]$ *where* $x \in \mathcal{X}$; $0 < y < t \leq T$;
$h_\theta(t,x) \; = \frac{\theta}{2} \inf_{u \in A} \Big[-r(t,x) - b(t,x)\,u \; + \; \frac{1}{2}(\frac{\theta}{2}+1)[u'a(t,x)u] \Big]$.
Then, $\psi_\theta(t,x,y)$ *is a mild solution of*

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial y}\right)\psi_\theta(t,x,y) + \frac{f(y \mid x)}{1-F(y \mid x)} \sum_{j \neq x} p_{xj}(\psi_\theta(t,j,0) - \psi_\theta(t,x,y))$$
$$+\frac{\theta}{2}\inf_{u \in A}\Big[-r(t,x) - b(t,x)u + \frac{1}{2}(\frac{\theta}{2}+1)u'a(t,x)u\Big]\psi_\theta(t,x,y) = 0$$
$$\psi_\theta(T,x,y) = 1.$$

(ii) ψ_θ *satisfies the following Volterra equation of second kind*

$$\psi_\theta(t,x,y) = \frac{1-F(T-t+y \mid x)}{1-F(y \mid x)} e^{\int_t^T h_\theta(s,x)ds} + \int_0^{T-t} \frac{f(y+\alpha \mid x)}{1-F(y \mid x)} \times \left(e^{\int_t^{t+\alpha} h_\theta(s,x)ds} \sum_j p_{xj}\psi_\theta(t+\alpha,j,0)\right) d\alpha. \quad (2)$$

Finite Horizon Case

Theorem

(iii) $\psi_\theta(t,x,y)$ *is the classical solution of the Cauchy problem.*
(iv) The risk sensitive optimal expected utility is given by

$$\varphi_\theta(t,v,x,y) := \sup_u J_\theta^{u,T}(t,v,x,y) = \log v - \frac{2}{\theta}\log(\psi_\theta(t,x,y)).$$

(v) **The optimal strategy for the risk sensitive criterion on finite horizon**
Case 1: $A = \mathbb{R}^n$, *the minimizing* $u_\theta^*(t,x)$ *in the PDE is given by*

$$u_\theta^*(t,x) = \frac{1}{1+\frac{\theta}{2}} a(t,x)^{-1} b(t,x)',$$

Case 2: *For A an n-dimensional rectangle, i.e.,* $A = \prod_{i\leq n} [c_i, d_i]$, *then*

$$u_{\theta,i}^*(t,x) = \begin{cases} \frac{1}{1+\frac{\theta}{2}} (a(t,x)^{-1} b(t,x)')_i & \text{if} \quad \frac{1}{1+\frac{\theta}{2}} (a(t,x)^{-1} b(t,x)')_i \in [c_i, d_i] \\ c_i & \text{if} \quad \frac{1}{1+\frac{\theta}{2}} (a(t,x)^{-1} b(t,x)')_i < c_i \\ d_i & \text{if} \quad \frac{1}{1+\frac{\theta}{2}} (a(t,x)^{-1} b(t,x)')_i > d_i \end{cases}. \quad (3)$$

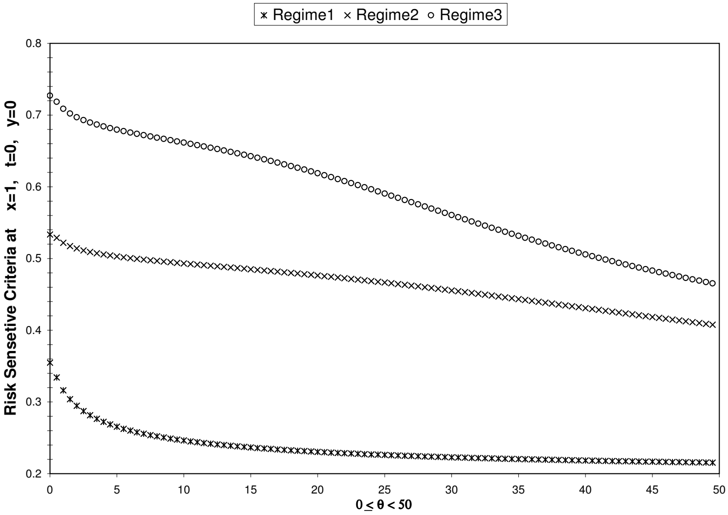

Figure : Risk Sensitive Optimal Expected Utility

Infinite Horizon Case: Optimal Strategy

We restrict ourselves to the autonomous model.

$$\frac{1}{T} J_\theta^{u_\theta^*, T}(v, x, y) \ \geq \ \frac{1}{T} J_\theta^{u, T}(v, x, y)$$

for any admissible strategy u. Hence

$$\liminf_{T\to\infty} \frac{1}{T} J_\theta^{u_\theta^*, T}(v, x, y) \ \geq \ \liminf_{T\to\infty} \frac{1}{T} J_\theta^{u, T}(v, x, y) \tag{4}$$

for all admissible strategy u. Thus u_θ^* is optimal for the respective action spaces for the infinite horizon case as well.

Optimal Value: Large Deviation Principle

If the limit exists

$$\lim_{T\to\infty} \frac{1}{T} J_\theta^{u_\theta^*,T}(v,x,y) = -\frac{2}{\theta} \lim_{T\to\infty} \frac{1}{T} \log\Big(E[e^{\int_0^T h_\theta(X_s)ds} \mid X_0 = x, Y_0 = y]\Big)$$

where

$$h_\theta(x) \; = \frac{\theta}{2} \inf_{u\in A} \Big[- r(x) - b(x)\, u \; + \; \frac{1}{2}(\frac{\theta}{2} + 1)[u' a(x) u]\Big].$$

Right side is the large deviation limit for semi-Markov process. For semi-Markov case the existence of the limit and the representation thereof in terms of some relative entropy is not available in the literature.

Subcase: Irreducible Markov State

$$E[e^{\int_t^T h_\theta(X_s)ds} \mid X_t = x] = \sum_{j=1}^{k} \exp\Big((\Lambda + diag(h_\theta(\cdot)))(T-t)\Big)(x,j).$$

Limit exists if the matrix $\Lambda + diag(h_\theta(\cdot))$ has a principal eigenvalue η (say). Then due to the multiplicative nature of the exponential function

$$\lim_{T\to\infty} \frac{1}{T} \log E[e^{\int_0^T h_\theta(X_s)ds} \mid X_0 = x]$$
$$= \lim_{T\to\infty} \frac{1}{T} \log \sum_{j=1}^{k} \exp\Big((\Lambda + diag(h_\theta(\cdot)))(T)\Big)(x,j) = \eta.$$

We prove the existence of η. Let $c := \min_x(\lambda_{xx} + h_\theta(x))$ and

$$\Lambda^\theta := \Lambda + diag(h_\theta(\cdot)) - cI_{k\times k}.$$

By Perron-Frobenius theorem the spectral radius of Λ^θ is an eigenvalue ρ. Therefore, $\eta := \rho + c$ is the principal eigenvalue of $\Lambda + diag(h_\theta(\cdot))$.

Subcase: Irreducible Markov State: Numerical Results

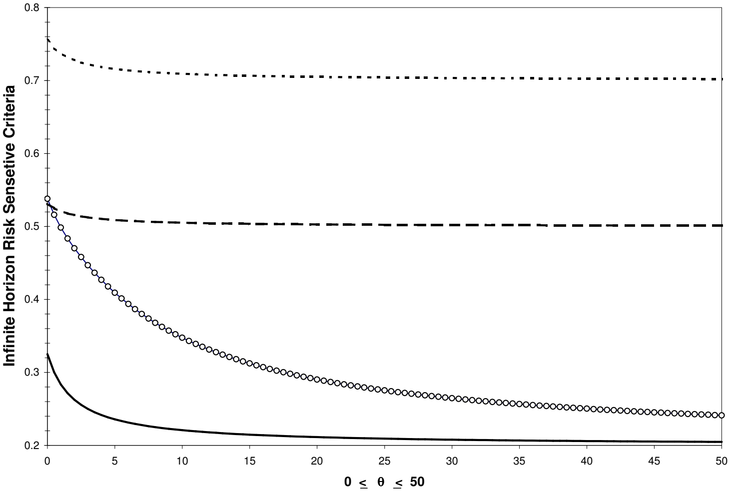

Figure : Infinite Horizon Risk Sensitive Optimal Portfolio Growth Rate

M. K. Ghosh, A. Goswami and Suresh K. Kumar, *Portfolio optimization in a semi-Markov modulated market*, Applied Mathematics and Optimization 60 (2009) 275-296.

Thank You