INTRODUCTION TO STATISTICS IN FINANCE

B.L.S. Prakasa Rao

CR RAO ADVANCED INSTITUTE OF MATHEMATICS, STATISTICS AND COMPUTER SCIENCE (AIMSCS)
University of Hyderabad CAMPUS
GACHIBOWLI, HYDERABAD 500046

e-mail:blsprao@gmail.com

STATISTICS IN FINANCE

- Risk is an important concept in the modern theory of finance. Every decision taken or transaction made by a company can be considered as the the buying or selling of risk. The success of a company is determined by how much profit it can make for a fixed amount of risk. Assets of an individual can be thought to be of two types; those which are risky and those which are risk-less.

STATISTICS IN FINANCE

- Shares owned by a person in a company can be viewed as an example of a risky asset as their total value fluctuates from one day to another depending on whether the price of the share moves up or down. Term deposits in a bank or government bonds are examples of risk-less assets in general as their future value depends on the interest to be earned which is known at the time of investment as a term deposit.

STATISTICS IN FINANCE

- A risk-less asset is an asset which has a clearly and precisely determined future value. A risky asset is an asset which is not risk-less. Risk can be considered as another way of looking at uncertainty in the decision making or in performing transaction.

STATISTICS IN FINANCE

- Almost any financial transaction is risky except for a risk less government bond or a term deposit in a bank. Another example of a risky asset is the amount of foreign exchange you are holding at a time point, say, in euros or in U.S. dollars. The future value in rupees of this holding in euros or dollars will depend on the fluctuations in the exchange rate between euros and rupees or dollars and rupees respectively.

STATISTICS IN FINANCE

- One important point to remember is that the investor in a company does not bother or worry about the riskiness of an asset if the cost of possible future value of the asset is more than its present value.

STATISTICS IN FINANCE

- The question is how to determine the present value of an asset. What do we mean by the value of an asset? Remember that, unless there is no inflation, an amount of thousand rupees a year from now will buy less than what we can buy with thousand rupees today; that is, the purchasing power in economic terms might go down as the time progresses.

STATISTICS IN FINANCE

- Value of an asset is what it can buy. The price of an asset depends on the risk taken in producing such an asset. In a free market or market driven economy, not controlled by government regulations, all the available information about an asset is already included in the price of an asset; hence there is no such thing as a good buy; the value of an asset depends on how much you and I are willing to pay to purchase the asset. The only value of an asset is its market value.

STATISTICS IN FINANCE

- Consider two people, one a professor in finance and the other an ordinary person, go on a walk and the ordinary person sees a currency note worth rupees one thousand on the street; when the person tries to pick up the note, the professor in finance says " Don't try to do that; it is absolutely impossible that there is a thousand rupee note lying on the street; for, if it were there, then some one else who passed through that way would have picked it up earlier". In other words, there is no free lunch.

STATISTICS IN FINANCE

- One of the basic assumptions in formulating models in the theory of finance is the concept of "no arbitrage". It essentially means that " no risk no profit". Everybody has to take risk if he or she wants to make a profit. Another way to put is it is that "there is no free lunch"!!

STATISTICS IN FINANCE

- Consider the trading of U.S. Dollars \$ versus Euros ϵ which takes place simultaneously at two stock exchanges, say, in New York in USA and Frankfurt in Germany. Suppose, for simplicity, that in New York, the $\$ - \epsilon$ rate is 1:1. Then it is obvious that the exchange rate, at the same moment of time, in Frankfurt should also be 1:1. Suppose, on the contrary, That you can buy one U.S. dollar in Frankfurt for Euro 0.999 euros. Then it is profitable to quickly buy U.S. dollars in Frankfurt and simultaneously sell the same amount of dollars for euros in New York and there by make a profit. This can be done on as large a scale as possible to increase profit. Such an opportunity is called an arbitrage opportunity.

STATISTICS IN FINANCE

- In such a case, the financial market cannot be in equilibrium or stable and the market forces triggered will make the dollars rise in Frankfurt and fall in New York. The arbitrage possibility will disappear when the two prices become equal in the sense that, even for arbitrageurs with very low transaction costs, the above scheme will not be profitable.

STATISTICS IN FINANCE

- An arbitrage opportunity is the possibility to make a profit in a financial market without risk and without net investment of capital. The principle of no arbitrage states that a mathematical model of financial market should not allow for arbitrage opportunities.

STATISTICS IN FINANCE

- In real financial markets, arbitrage opportunities can and do exist. But they will be present only for a short amount of time and disappear quickly as some one will always be ready to use whenever they appear. In Mathematical theory of finance, it is therefore always assumed that the market is functioning under no arbitrage or in other words there are no arbitrage opportunities.

STATISTICS IN FINANCE

- The job of a statistician is to develop methods to study market behaviour under an arbitrage free market. Assuming that all the assets are correctly priced by the market, the question is how to differentiate one asset from the other to decide which one to buy or which one to sell.

STATISTICS IN FINANCE

- An important component a market has is its information on the riskiness of an asset . This information is reflected in the pricing of an asset. Increased risk means greater returns on the average which also means possible greater losses. An asset's price reflects the value it is likely to have in the future taking into account its riskiness.

STATISTICS IN FINANCE

- It is not possible to predict the future price of an asset from past data in general. Past information on an asset is already included in its present price. The purpose of modeling in stochastic finance is not for prediction of the prices of several types of assets but to correlate the movements of the price of one asset to that of another.

STATISTICS IN FINANCE

- The price movements are considered to be driven by the information arriving/available about an asset in the market and this information is unknown and can be considered as random. The main issue in mathematical finance is to construct a portfolio, that is, a combination of the market instruments which are risky and risk-less affected by the same information, to reduce or remove or cancel randomness. This process is known as *hedging*. The objective is to develop some methods to hedge and to understand its consequences.

STATISTICS IN FINANCE

- Let us now look at two types of assets available in the market one of which is risky and the other risk-less. Consider a government bond issued by the Reserve bank of India for instance. The government bond is a bond, say, for 5 or 7 years or more in term, and which pays either every year an interest on the bond and gives the investor his or her original deposit invested at the end of the term or pays the total interest accumulated over the bond period along with the principal at the end of the maturity of the bond.

STATISTICS IN FINANCE

- This type of investment is a risk-less investment as the future value of the investment is known at the time of initial investment. In contrast to this type of investment, suppose you bought a share or stock in a company. Future value of the stock or share depends on the market behaviour. It might be more or less than its present value. Such an investment is an example of a risky investment.

STATISTICS IN FINANCE

- Let us look at the investment such as stocks or shares in a company in more detail. The holder of a share of a company owns a part or a fraction of the company. The word " Limited" in the name of the company signifies that the share holder has limited liability. Another asset commonly traded in the market is a company bond or a corporate bond. Riskiness for such a bond is generally higher than that associated with a government bond but lower than the risk associated with a stock or a share of a company.

STATISTICS IN FINANCE

- A company which needs a loan for raising its capital might issue bonds in market by paying interest. The rate of interest is generally higher than that of a risk-less bond issued by the government. The investor's risk is that the company may default in its payment of interest. However the holder of the company bond has higher claim on the company's assets than the share holders and hence the riskiness for the share holders is lower. Companies also issue bonds at times known as the debentures which may be convertible indicating that they can be converted into the shares of the company after the maturity of the bonds depending on the share value at the time of convertibility.

STATISTICS IN FINANCE

- All the type of assets discussed till now such as government bonds or corporate or company bonds are similar in the sense that the value addition to them after maturity is either positive or zero. However there are other assets which might lead to negative values or losses to the values of assets. We will discuss more about such assets later. There is another market instrument known as option which is being traded in the market.

OPTIONS

- In order to explain what an option is, let us consider the following scenario. Every country is dependent on its energy resources, such as oil, in particular, petroleum. For instance, India is a vast country of 1.2 billion people and the demand for oil is increasing everyday. India does have its own resources but the output from them is not sufficient for its needs. India has to import oil from other countries by paying in foreign exchange. The price per barrel of crude oil is decided by the Organization of Petroleum Exporting Countries (OPEC). India has to make plans well ahead, at least an year in advance, to estimate its production and to estimate the amount of oil it needs to import. If it does spot buying, then it may have to pay an exorbitant rate per barrel. However if it buys an option to buy, say, one million barrels at a specified rate at a specified time, then it can control its expenses.

OPTIONS

- This is the idea behind options. Let us now look at this in more detail. Given that risk is inherent in all decision making processes either by a bank or by a company or by a government of a country, it is important to see whether such a risk can be managed. A bank or a company may buy any one or more of several market or financial instruments which are likely to decrease its risk. Option is one such an instrument. It can be used to reduce the risk but if is misused , then it might increase the risk. Diversification is another way of reducing the risk by a company.

OPTIONS

- As the saying goes "Do not place all the eggs in one basket". For if you do and the basket is dropped, then you will loose all of your investment. In recent times, companies, such as those involved in manufacture of tobacco products such as cigarettes, have started diversification of their products such as into manufacture of biscuits or into manufacture of writing or packing paper or administration of big hotels etcetera to minimize the risk, as they have noticed that consumption of tobacco products is declining among the public leading to reduction in profits for the companies.

OPTIONS

- What is an option? An option is an instrument which gives the holder the right to buy or sell the quantity of some fixed asset during a specified time period, called strike period, at a price fixed today, called the "strike price" or "Exercise price". If the holder is planning to buy an option, it is called a "call option"; if he is trying to sell an option, it is called a "put option".

OPTIONS

- The holder of the option has the right but not obligation to buy or sell depending on the type of option. An option is called an "European Option" if the holder of the option can exercise his option only at the specific time known as the "Exercise time" or "strike time" and not before. It is called an "American option" if the holder of the option can exercise his option only at any time before or at the specific time known as the "Exercise time" or "strike time".

OPTIONS

- There are also other type of options which we will not discuss now. The value of an option is sensitive to market fluctuations and hence the amount to be gained or lost by exercising an option are large. The purpose of an option is to allow the buyer or seller of the option to guard against certain unforeseen events which might be catastrophic and thus reduce the risk.

OPTIONS

- The difference in buying a call option by a holder of a stock or buying the stock is that if the stock price goes the wrong way, the option will have no value for the buyer as it will not be exercised and if the stock price moves goes the right way, the option will be exercised and the holder will make a positive profit. How to price such an option?

OPTIONS

- Participants in markets activities can be classified into different categories. Some are "hedgers". These use market instruments to reduce risks. For instance, they might choose a portfolio, a combination of risky assets such investments in stocks, options on stocks and term deposits in banks or government bonds which are generally risk-less, to minimize the risk.

OPTIONS

- The second group may be termed as "speculators." They use market instruments to increase their profits by preparing to take large risks if necessary and in the process might incur heavy losses.

OPTIONS

- The third group come under the category of "arbitrageurs". This group tries to find discrepancies in the pricing of risks and adjust their portfolio to make profit without risks. Companies are hedgers; banking institutions are a combination of speculators and arbitrageurs and private investors are generally speculators.

OPTIONS

- Let us look at some possible methods for determining the price of a European call option. Suppose the price of a stock (share price) today is Rupees 95 and we are interested in buying a European call option on the stock with a "strike price" of Rupees 200 and "strike time" 5 years from today. How much should I pay today for purchasing this option?

OPTIONS

- Suppose the stock price will be Rupees 300 five years from today. If the option price today is fixed at a price, say Rupees 150, more than the stock price today, then the seller of the option can buy the stock from the market at a lower price of Rupees 95 and sell the option at Rupees 150/= and make a profit of Rupees 55 without any risk to him or her. It is clear that there is an arbitrage opportunity for the seller if the option price today exceeds the stock price today irrespective the price of the stock five years from today.

OPTIONS

- Since we consider only those models without arbitrage opportunities, it follows that option price should never exceed the stock price on the day of purchase of option as otherwise there will be arbitrage opportunities. There might be other reasons for an option seller to fix the option price at a particular level.

OPTIONS

- The person may want to have a good return on the average or bound the total amount that might be lost or minimize the riskiness of the outcome or invest that much of an amount today that will cover the cost of option payoff on the strike date or avoid misspecifying the risk. Any one of these might dictate in fixing the option price for the option seller. The first objective is that of a speculator while the others are those of hedgers and arbitrageurs. The main aim of statisticians in finance is to find possible prices of options and other market instruments in order to achieve the last four objectives as much as possible under no arbitrage opportunity.

ASSUMPTIONS

- In order to build mathematical models for finance, we assume
  (i) Actions such as buying or selling of a stock do not affect the market price, that is, any one can buy or sell any amount of a particular stock. This assumption in toto might not hold in a free market since the demand and supply are closely related. If the demand increases, then the price increases encouraging production and if the demand decreases, then the price decreases discouraging production. Thus the action of buying or selling can affect the market price. However, if the trading is in small quantities, the effect will be negligible.

ASSUMPTIONS

- (ii) We assume that there is enough liquidity in the market in the sense that one can buy or sell at any time as much as we wish at the market price. This assumption is not fully valid, for instance, in the foreign exchange market in India at the present time. Indian currency is partly convertible at the present time. However the assumption holds in some commodity markets. Buying and Selling by speculators and other traders increases the liquidity in the market.

ASSUMPTIONS

- (iii) We assume that one can trade, that is , buy or sell, fractional quantities of assets. This might not be possible for individual traders dealing in shares in small quantities but for traders such as banks involved in foreign exchange transactions in millions of units, such an assumption is valid.

ASSUMPTIONS

- (iv) We assume that there are no trading costs, that is, one can buy or sell without any transaction cost. This is not true in any market as transactions do involve costs. Our assumption simplifies the models.

ASSUMPTIONS

- (v) It is also assumed that "shorting" is possible, that is one can have negative amount of assets by selling assets which one does not hold (some times called "go short") at will. Similarly buying an asset is called "going long". This is not possible in some countries.

ASSUMPTIONS

- These are the basic assumptions, in addition to the "no arbitrage assumption", under which methods of modeling in finance are developed.

PRESENT VALUE ANALYSIS

- What is meant by the value of an asset? It is obvious that if you have Rupee.1000/= one year from now will buy less than what it does today unless there is no inflation, that is unless there is no price rise. The cost of an apartment today might be much less than what it would cost next year with a possible price rise in components such as steel. Cars today will cost more in future than what the same costs today. The value of an asset depends at what time you are inquiring about it. If the present value of a bond is Rupees 10,000/=, a year from now its value increases due to the interest earned on Rupees 10,000/= over a period of one year. The interest on the bond might be a simple or it might be compounded monthly.

PRESENT VALUE ANALYSIS

- Suppose I have an amount of P rupees, called the Principal here after and I will deposit in a bank for a term period of T years. Suppose the bank pays me a simple interest at the rate of $r\%$ per year. What would be the value of my term deposit (TD) at the end of one year? It is obvious at the end of the first year, the value of the TD will be the principal plus the interest accrued over a period of one year, that is, $P + Pr = P(1 + r)$ rupees. At the end of the first year, we note that the value of the TD is $P(1 + r)$ which will act as the principal amount. At the end of the second year, the value of TD will be $P(1 + r)(1 + r) = P(1 + r)^2$ rupees. Proceeding in this way, we can show that the value of the term deposit at the end of T years will be $P(1 + r)^T$ rupees. Hence the value of an investment of Rupees P at the end of T years is $P(1 + r)^T$ rupees if the rate of simple interest is $r\%$ per year.

PRESENT VALUE ANALYSIS

- Let us again suppose that we have borrowed P rupees from a bank which has to be returned to the bank at the end of one year. Suppose the bank charges an interest rate of $r\%$ per year compounded at n equal intervals of time during one year. Following the same arguments as given earlier, it is obvious that the total amount to be repaid will be
$$P(1+\frac{r}{n})^n.$$
Let $n \to \infty$. In such a case, we say that the interest rate is *compounded continuously*. If the interest rate is compounded continuously, then the total amount to be paid to the bank will be $\lim_{n\to\infty} P(1+\frac{r}{n})^n = Pe^r$. If the amount is to be paid along with the interest compounded continuously at the end of T years, then the total amount to be paid will be Pe^{rT} rupees.

PRESENT VALUE ANALYSIS

- If P rupees is the value of a TD at the end of T years and the interest rate r is compounded continuously, then its present value is Pe^{-rT}. If the interest rate is compounded yearly, then its present value is $P(1+r)^{-T}$.

RETURNS

- Let us consider an investment of Rupees $x > 0$ in a company A which fetches a return of Rupees y after one period, say, an year. Then the rate of return r of the investment is defined to be that value r such that the present value of the return y after one year is equal to x the amount invested now, that is, $\frac{y}{1+r} = x$ or equivalently $x(1+r) = y$. Therefore

$$r = \frac{y-x}{x}.$$

If the return $y \geq 0$, then the rate of return $r \geq -1$.

RETURNS

- Let us now look at the returns from the company A at the end of n consecutive time periods, say, at the end of over n years. Let y_i be the return at the end of i-th year. The flow of returns is $\mathbf{y} = (y_1, \ldots, y_n)$ over n years. Suppose r is the rate of interest compounded yearly. Then the present value of the cash flow $\mathbf{y}$ of returns is

$$PV((\mathbf{y}) = \frac{y_1}{1+r} + \cdots + \frac{y_n}{(1+r)^n}.$$

and the return $P(r)$ at the end of n years is

$$P(r) = \frac{y_1}{1+r} + \cdots + \frac{y_n}{(1+r)^n} - x.$$

RETURNS

- Note that return on an asset such as stock is the ratio of change in prices to the initial price, that is, if P_t is the price of a stock at time t, and the stock was held during the time $t-1$ to time t, then the return is

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}}$$

assuming that no dividend was paid by the company during the period $[t-1, t]$. It is obvious that R_t may take negative values but $R_t \geq -1$ since P_t is nonnegative for all $t \geq 0$. The function R_t is also called "net return" of the stock at time t. and the function

$$\frac{P_t}{P_{t-1}} = 1 + R_t$$

is called "gross return" of the stock at time t.

RETURNS

- Returns at time t are independent of the units of measurement such as rupees or any other unit used for pricing the stocks but they depend on the time t of measurement. Let

$$r_t = \log \frac{P_t}{p_{t-1}} = \log(1 + R_t).$$

The quantity is called "log return" of the stock at time t. If R_t is small, then the net return R_t and the log return r_t are almost the same.

LOG RETURNS

- Let us now look at the net return $R_t(k)$ of a stock at time t, starting from k units of time before time t defined by

$$R_t(k) = \frac{P_t - P_{t-k}}{P_{t-k}}.$$

Then, for $t \geq k$,

$$1 + R_t(k) = (1 + R_t)(1 + R_{t-1})\ldots(1 + R_{t-k+1})$$

and hence

$$\log(1+R_t(k)) = \log(1+R_t)+\log(1+R_{t-1})+\cdots+\log(1+R_{t-k+1}).$$

LOG RETURNS

- Let $r_t(k) = \log(1 + R_t(k))$ be the "log return" of a stock at time t, starting from k units of time before time t. From the elementary property of the logarithmic function,

$$r_t(k) = r_t + r_{t-1} + \cdots + r_{t-k+1}.$$

  Hence the log return over k periods before time t, that is over the time period $[t-k, t]$ is the sum of log returns over the periods $[t-k, t-k+1], [t-k+1, t-k+2], \ldots, [t-1, t]$. This shows the "additivity" property of the function "log returns".

DISTRIBUTION OF LOG RETURNS

- An important problem is to model the probability distribution of the sequence of returns $R_1, R_2, \ldots$ of a stock over consecutive periods of time. These random variables are possibly neither independent nor identically distributed and they take values in the interval $[-1, \infty)$ and hence can not be modeled by Gaussian random variables. However the log returns $r_1, r_2, \ldots$ take values in the real line and they satisfy the additivity property. As the support of the Gaussian distribution is the real line and the sum of independent Gaussian random variables is also a Gaussian random variable, one model that was suggested is to assume that the log returns form i.i.d. Gaussian random variables with some mean and some variance.

LOG-NORMAL DISTRIBUTION

- A positive random variable X is said to have a log-normal distribution if the random variable $Y = \log X$ has a normal distribution. Check that

$$\mu_X \equiv E(X) = E(e^Y) = e^{\mu+\frac{1}{2}\sigma^2}$$

and

$$\sigma_X^2 \equiv Var(X) = e^{2\mu+\sigma^2}[e^{\sigma^2} - 1].$$

LOG-NORMAL DISTRIBUTION

- Log-normal distribution is a heavy tailed distribution and is asymmetric. A measure of skewness of a random variable X is
$$\frac{E(X - E(X))^3}{[Var(X)]^{3/2}}$$
whenever it exists. The skewness does not depend on the location or scale parameter. If the distribution is symmetric, then the skewness is zero. If the skewness is positive, then it indicates that the distribution has a heavy right tail as compared to the left tail and if the skewness is negative, then the distribution has a heavy left tail as compared to the right tail. The left tail of a distribution is the region $(-\infty, \mu_X - 2\sigma_X)$ and the right tail is the region $(\mu_X + 2\sigma_X, +\infty)$.

LOG-NORMAL DISTRIBUTION

- Distributions with high tail probabilities compared to the Gaussian distribution with the same mean and variance are called *heavy tailed*. Heavy-tailed distributions are suitable for modeling in finance as the stock return distributions have been observed to have heavy tails. Check that the skewness of the log-normal distribution with parameters μ and σ^2 is

$$[e^{\sigma^2} + 2](e^{\sigma^2} - 1)^{1/2}.$$

LOG-NORMAL DISTRIBUTION

- From the representation of the $\log(1 + R_t(k))$ over k periods, we get that $\log(1 + R_t(k))$ has the normal distribution with mean $k\mu$ and variance $k\sigma^2$. Hence the skewness of the return $R_t(k)$(or equivalently of $1 + R_t(k)$) is

$$[e^{k\sigma^2} + 2](e^{k\sigma^2} - 1)^{1/2}.$$

which is positive and increases to infinity rapidly as $k \to \infty$. Hence if the log returns over one period are normally distributed, then the distribution of returns, when the stock is held over a long period,is highly skewed.

LOG-NORMAL DISTRIBUTION

- There are different methods to see whether log-normal distribution is a good-fit for the distributions of log-returns. One method is to look at the corresponding normal probability plot. The normal probability plot is a graph of the sample quantiles against quantiles of the standard Gaussian distribution. If the plot is almost a straight line, then it indicates that the sample of log returns is likely to be from a Gaussian distribution. Another way to check normality is by computing the skewness and kurtosis of the log returns, Recall that the kurtosis of a random variable Y is

$$\frac{E(X - E(X))^3}{[Var(X)]^2}.$$

For a Gaussian distribution, skewness is equal to zero and kurtosis is equal to three. We might caution the skewness and kurtosis are very sensitive to outliers if any in the data.

RANDOM WALK

- We have seen that a suitable model for the distribution of the log-returns is the log-normal distribution and the log-returns over k-periods is the sum of the log-returns over the individual periods. This is an example of random walk model. Suppose $\{Z_1, i \geq 11\}$ are independent and identically distributed (i.i.d.) random variables with mean μ and variance σ^2. Let S_0 be another random variable independent of the sequence $Z_i, i \geq 1$. Let

$$S_n = S_0 + Z_1 + \cdots + Z_n, n \geq 1.$$

  The stochastic process $\{S_n, n \geq 1\}$ is called a *random walk*. Observe that

$$E(S_n|S_0) = S_0 + n\mu.$$

  and

$$Var(S_k|S_0) = n\sigma^2.$$

RANDOM WALK

- The sequence $\{S_n, n \geq 0\}$ may be interpreted as the the sequence of stock prices or share prices with S_k as the price at time k with S_0 as the intial price. The parameter μ is called the *drift* which indicates the trend in the stock prices and the parameter σ is called the *volatility* indicating the fluctuations from the average price. Volatility is a measure of how much the random walk fluctuates from the drift or trend μ. For a stock broker, the drift μ of a stock price is not the main issue as he is or she is aware of the same from the past information but the volatility σ is.

RANDOM WALK

- If the volatility is high, there is a likelihood of heavy gains or heavy losses due to large fluctuations in the stock prices. Suppose the drift μ and the volatility σ are known or can be estimated accurately. Recall that the log-return over k periods before time t is

$$r_t(k) = r_t + r_{t-1} + \cdots + r_{t-k+1}$$

where $r_t, r+_{t-1}, \ldots, r_{t-k+1}$ can possible be considered as i.i.d. random variables. Let P_t denote the stock price at time t. Then the return over consecutive k periods before time t, is

$$R_t(k) = \frac{P_t - P_{t-k}}{P_{t-k}} = \frac{P_t}{P_{t-k}}$$

and

$$\frac{P_t}{P_{t-k}} = 1 + R_t(k) = e^{r_t(k)} = e^{r_t + \cdots + r_{t-k+1}}.$$

RANDOM WALK

- Let $k = 0$. This shows that

  $$\frac{P_t}{P_0} = e^{r_t + \cdots + r_1}$$

  or

  $$\log P_t = \log P_0 + r_t + \cdots + r_1 .$$

  This shows that the process $\{\log P_t, t \geq 0\}$ can be considered as a random walk. Then the process $\{P_t, t \geq 0\}$ is called a *geometric random walk*. Under this model, the changes in the stock prices in future are independent of the past and hence not predictable.

RANDOM WALK

- Let us assume that the distribution of the log-returns is $N(\mu, \sigma^2)$ independent of the initial price P_0. Under this model, let us compute the mean and variance of the stock price P_t given that the initial price of the stock is P_0. Then

$$E(P_t|P_0) = P_0\ [e^{t\mu + \frac{1}{2}t\sigma^2}]$$

and

$$Var(P_t|P_0) = P_0^2\ Var(e^{r_1 + \cdots + r_t}|P_0).$$

from the properties of the log-normal distribution.

RANDOM WALK

- Check that the conditional distribution of the random variable P_t given P_0 is log-normal with parameters $\log P_0$ and $t\sigma^2$. Under this model, check that the median price of the stock at the end of t periods is $P_0\ e^{t\mu}$ and the mean price is $P_0\ e^{t\mu+t\frac{\sigma^2}{2}}$.

OPTION PRICING

- An *option* gives one the right, but not obligation, to buy or sell a specified number of shares of a stock at a special price called the *exercise price* or the *strike price* on or before a specified future date called the *date of maturity* or *strike date* or *expiration date* or *exercise date*. The option is called a *call option* if the option is to buy a stock and it is called a *put option* if the option is to sell a stock.

OPTION PRICING

- A *European call option* gives a person the right, but not obligation, to purchase a specified number of shares of a stock at a specified rate called the strike price on a specified date called the date of maturity. It is called a *European put option* if it gives a person the right, but not obligation, to sell a specified number of shares of a stock at a specified rate called the strike price on a specified date called the date of maturity. There are other types of options such as an American call option and an American put option.

OPTION PRICING

- An *American call option* gives a person the right, but not obligation, to purchase a specified number of shares of a stock at a specified rate called the strike price on or before a specified date called the date of maturity. It is called a *American put option* if it gives a person the right, but not obligation, to sell a specified number of shares of a stock at a specified rate called the strike price on or before a specified date called the date of maturity.

OPTION PRICING

- In order to purchase such options, a price has to be paid called the option price. How do we determine, for instance, the price of an European call option, given the strike price and strike date?

OPTION PRICING

- Let us consider the following scenario. Suppose that we have purchased a European call option for 100 shares of stock A with an exercise price of Rupees 80 per share and the expiry date six months from today. On the expiration date, suppose the stock is selling at Rupees 85 per share.

OPTION PRICING

- The option allows us to purchase 100 shares for Rupees 80 each on the date of expiry and *immediately* sell them on the same day for Rupees 85 each at the market price and there by make a gross profit of Rupees 5 per share for 100 shares, that is, a profit of Rupees 500.

OPTION PRICING

- Since we have to pay a price or a premium for purchasing the option, the net profit on the date of expiry of option is not Rupees 500. If we have paid Rupees 3 per share for purchasing the option, then the cost of purchase of option to buy 100 shares will be Rupees 300.

OPTION PRICING

- Suppose the interest rate is 10% per year compounded continuously. The interest rate will be 5% for a period of 6 months compounded continuously and the present value (today's value) of the net profit will be

$$e^{-.05}(500) - 300$$

and its value on the date of expiry of the option is

$$500 - (300)e^{.05}.$$

OPTION PRICING

- Note that a call option is never exercised if the strike price is larger than the stock price on the date of expiry. This is obvious since exercising the option leads to purchase of the stock at a price higher than the strike price and the option holder will incur loss. If a call is not exercised, then the option holder might still loose some money due to the cost involved in the purchase of the option. It is possible that one can still loose money on an option even if it is exercised since the amount gained by exercising the option might be less than the premium or the price paid for purchasing the option.

OPTION PRICING

- Let S_t be the stock price or share price of a stock at the time t. Let K be the exercise price and T be the expiry time for a European call option. If $S_T \geq K$, the we exercise our option and buy the stock at a price of K per share and then sell the stock immediately on the same day to make a gross profit of $S_T - K$ per share. If $S_T < K$, then we do not exercise the option and no profit is made.

OPTION PRICING

- Let $x_+ = x$ if $x \geq 0$ and $x_+ = 0$ if $x < 0$. The profit to be made on the date of expiry is $(S_T - K)_+$. If r is the interest rate compounded continuously, then the present value of the profit $(S_T - K)_+$ is

$$e^{-rT}(S_T - K)_+.$$

OPTION PRICING

- Since we do not wish to incur loss by purchasing the option, the cost of the option or the option price C should be $E[e^{-rT}(S_T - K)_+]$ and hence

  $$C = E[e^{-rT}(S_T - K)_+]$$

  where the expectation is computed with respect to a suitable probability measure under the no arbitrage assumption and assuming that the trading plan is self-financing.

OPTION PRICING

- A trading plan or strategy is said to be *self-financing* if it requires no investment from outside sources other than the initial investment and if it allows no withdrawals. After the initial investment, additional purchase of assets are made from the sales of other assets or by borrowing and the proceeds of any sale of an asset is always reinvested. Recall that an arbitrage opportunity indicates that one can make a guaranteed risk-free profit by trading in the market. We call the price of a market instrument, such as an option, as the arbitrage price of the option if it is the price which guarantees no arbitrage opportunities under the self-financing plan.

OPTION PRICING

- We assume that our trading is self-financing with no arbitrage opportunities. This leads to an important law known as *the law of one price*.
  The law of one price: If two financial instruments have the same payoffs, then they should have the same price.

OPTION PRICING

- The law of one price will be used to find the option prices. To find the value of an option, we will construct a portfolio or self-financing plan with a *known price* that has exactly the same payoff as the option. Then, by the law of one price, the price of the option should be the same as the price of the portfolio under the self-financing plan with no arbitrage opportunity.