Options and Derivative Pricing

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The slides are based on the following:

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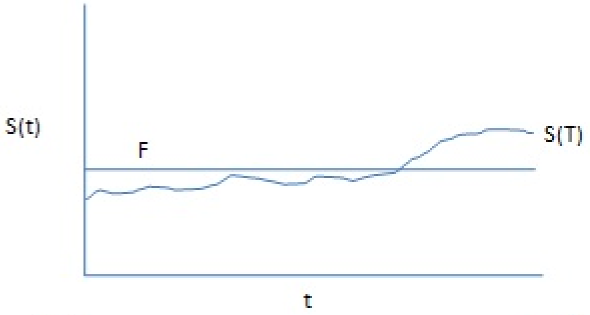
Derivatives

- A financial derivative is a financial product which is derived from an underlying asset such as stocks of a company.
- Its payoff explicitly depends on the price of the underlying asset.
- Are traded when individuals or companies wish to buy asset or commodity in advance to insure against adverse market movements.
- Provide effective tools for hedging risks.They are designed to enable market participants to reduce risk.
- Simplest example of a derivative is the *forward*: individuals enter into a contract to buy an asset at some fixed time in the future for a certain price fixed today. Other examples are: Futures contracts, Options.



t = 0

The buyer enters in to a contract with the seller to buy the asset at time **T** for a price **F.** Market price of the asset = S(0).

t = T

The buyer buys the asset from the seller for price F. Market price of the asset: S(T).

Futures

A future is a standardized forward contract. A regulatory body such as a Stock Exchange (when the underlying asset is a stock), is involved. There is typically a set size of the contract.
The aim is to determine a fair value for **F**.

Options

The buyer/holder of an **option** has the right, but not the obligation, to do something like buy or sell an asset (stock) for a fixed price up to a certain fixed time in the future. The seller is obligated to carry out the contract. Options are traded on exchanges and the exchange specifies certain standard features of the contract. Options come in many different types of which the most important and which are also traded on NSE of India are the American and European options.

Options can be on stocks, stock indices, currencies and futures and commodities.

Options on stock indices are settled in cash.

- - A **European call option** is a contract between two parties giving the buyer (holder) the right to buy an asset from the seller (writer) of the option at a fixed price K, called the strike price, at time T called the expiry or the terminal time.
  - The *European put option* gives the buyer the right but not the obligation to sell the asset for a strike price K at a fixed time T in the future.
  - The **European** refers to the fact that the option is exercised at the terminal time. An **American option** is one in which the option can be exercised at any time up until the terminal time T.
- **When one buys an option, one has to pay a certain initial price (premium) to the seller. The aim is to determine the 'fair' price.**

1.**Hedger**: Person who buys options as a form of insurance against adverse market movements.

Example: An investor owns 100 shares of some company and wants to protect against a possible decline in the next one month. Current price of the share = Rs 855; price of a Jan.27 American put = Rs 15.00; Strike price = Rs 860. Strategy: The investor buys one American Put option contract (a contract consists of 100 shares) for a total cost of Rs $15 \times 100 = 1500$ Rs.

Outcome: The investor has the right to sell 100, shares for at least $(100 \times 860) = 86000$ Rs. on or before Jan. 27

If the price remains above Rs 860, the investor need not sell (at 860) . If the price goes below Rs 860, the investor can sell for Rs. 860. This way the investor knows that the **maximum loss** he could suffer by Jan.27 = 1500 + (85500 - 86000) = Rs. 1000.

2.**Speculators**: the individuals who gamble on the way various assets will move. They take a position in the market based on their beliefs and make or lose money based on (essentially) chance. They are interested in financial derivatives because it is possible to make a lot more money by buying or selling these products rather than the underlying asset.

Example: An investor has Rs 4000 to invest. He feels that the price of the share of company A will increase in the next two months. Suppose the current price = Rs 200,
Jan. 30 European call price (premium) = Rs 8;
Strike price = Rs 220.
Strategy 1: Buy 20 shares.
Strategy 2: Buy 500 call options (5 contracts).

Possible Outcomes: a)Suppose the share price rises to Rs 250 by Jan. 30.
Strategy 1 Profit = (5000 - 4000) = 1000 Rs.
Strategy 2 Profit = (500(250 - 220) - 4000) = 11,000 Rs. (The buyer has the right to purchase the Rs 250 share for Rs 220 only. Thus he can purchase a share for Rs 220 and sell for Rs 250 and make a profit of Rs 30 per share).
b)The share price falls to Rs 180 by Jan. 30. Strategy 1 Loss = 20(200 - 180) = 400 Rs.
Strategy 2 Loss = Rs 4000. (The buyer will not buy the asset and loses the initial investment.)

3. **Arbitrageurs**: People who watch the market and try to find situations where there are risk free profits to be made.

Example: Suppose the current price of a share is Rs 200. Price of a European call option is Rs 20; Strike price $=$ Rs 180; Expiry date $=$ 1 year. Suppose it is possible to invest money (say in a bank) with a risk free interest of 10% per annum.

Strategy: **Short** sell the share for Rs 200. With this money buy the option for Rs 20 and invest Rs 180 at 10% p.a. for one year.

(**Note:** Short selling of shares, that is selling shares one does not own, is permitted. An investor can instruct a broker to short sell certain number of shares of a company. The broker will carry out the instructions by borrowing the shares from another client and selling them in the market in the usual way. At some stage the investor will close out the position by purchasing the same number of shares of the same company and these are replaced in the account of the client from whom the shares are borrowed.)

If at the end of one year, the share price is > 180, buy the share and close the short position (return the borrowed share). Money obtained from investment in the bank $= 180 \times 1.1 = 198$. Thus Profit $=$ Rs 18 (per share.)

If the price goes below Rs 180, do not exercise the option. From the Rs 198 obtained from the bank, buy the share with price < 180 and close the short position. Thus Profit $\geq 198 - 180 = 18$ Rs.(per share).

Thus whatever the share price, the investor has made a profit without taking any risk on his own assets.

Assumptions

- A basic assumption that is made while determining the fair price of an option is that there are no arbitrage opportunities. An arbitrage opportunity is a strategy which will return a positive amount with certainty, often described as 'a Free Lunch'.
- There is a risk less asset called a bond (a zero-coupon bond). Think of this as money placed in an account at the Bank. It accrues interest in continuous time at a fixed rate r. So if B_0 is the amount of money invested in the bond (bank) at time 0, then at time t the value of the bond is $B_t = B_0 e^{rt}$. Thus the *present value* of a payoff of Rs. B to be made at the time period T is Be^{-rT}.

$(B_t = B_0(1 + r/(m))^{mt},\ m \to \infty)$

Assumptions

1. No transaction costs.
2. Borrowing and lending at the same risk free rate r is possible.
3. One can trade in fractional quantities of the asset.
4. There is liquidity in the market and one can buy or sell at any time.
5. All trading profits are subjected to the same tax rate.
6. No dividends on the underlying stock. If there is to be a dividend on the stock during the life of the option, then the initial stock price is considered to be $S(0) - D$, where D denotes the present value (at time=0) of the dividend.
7. Short selling is possible.

Forward contract

To determine the price F that should be agreed to now for payment in the future at time T.

Suppose the current price of the asset is $S(0)$.

Consider the following strategies.

Strategy 1

If $F > S(0)e^{rT}$, the investor can sell the forward contract (i.e. agree to sell a unit of asset to the other party at time T) for F, then borrow enough money, i.e. $S(0)$ to buy one unit of the asset and hold on to the asset until time T. At time T, the investor has the asset, he can exchange it with the other party for F. He has to pay back the money borrowed which amounts to $S(0)e^{rT}$. His net profit is $F - S(0)e^{rT}$.

Strategy 2

If $F < S(0)e^{rT}$, the investor should take a long position in the forward contract for F, then he sould borrow the asset from a friend, sell the asset for $S(0)$, invest that money in the bank up to time T. At time T, he will get $S_0 e^{rT}$, use it to buy the asset for F and then return the asset to the friend. The profit in this case is $S(0)e^{rT} - F$.

Thus we see that unless the price of the forward is fixed at $F = S_0 e^{rT}$ then by following either strategy 1 or strategy 2, there is a risk free way of making profit. This would be an arbitrage opportunity and hence the price of the forward is fixed by the necessity to avoid arbitrage. Note that the fair price is not $E_P[S(T)]$, where P is the probability law according to which the prices evolve.

European call and put stock options

Let the prices (premiums) be c and p, respectively.

- current stock price $= S(0)$
- strike price $= K$
- expiry time $= T$,
- the risk free interest rate $= r$.

Then simple lower bounds (without assuming any model) on the prices c and p so that there is no arbitrage opportunity are

$$\max\{S(0) - Ke^{-rT}, 0\} < c \quad \text{and} \quad \max\{Ke^{-rT} - S(0), 0\} < p.$$

put-call parity:

$$c + Ke^{-rT} = p + S(0).$$

Proof using the no arbitrage principle:
Suppose $c + Ke^{-rT} > p + S(0)$.
Strategy at time t = 0: sell one call, borrow Ke^{-rT}, buy one put and one share. Then the profit is: $c + Ke^{-rT} - (p + S(0)) > 0$.
Strategy at t = T:

- If $S(T) > K$, the buyer of the call option will exercise his right and buy the share at K,
- you can use this amount to return the borrowed cash with interest, i.e. return an amount K.
- The put will be worthless.
- Thus the cash flow is $K - K = 0$, and there is no loss.

If $S(T) \leq K$:

- the buyer of the call option will not buy the share.
- You can use the put option and sell the share for an amount K.
- You can use this amount to return the borrowed cash with interest, i.e. return an amount K.
- Thus the cash flow is $-K + K = 0$, and there is no loss.

Thus there is profit at $t = 0$ and no loss whatever happens to the share price.

This is arbitrage. Therefore $c + Ke^{-rT} \leq p + S(0)$.

The reverse strategy may be used if $c + Ke^{-rT} < p + S(0)$.

In order to determine the pricing of this contract, the first step is to model the behaviour of the underlying. We will consider options on stocks and model the the stock price as a Markov process. In a Markov process the future movements in a variable depend only on the present, and not the history of how we got to the present value. In fact we will see that we only need to know how the price will move and not the actual probabilistic structure!

A Binomial Model or Binomial Tree for the Stock Price Dynamics (Cox-Ross-Rubinstein model (1979))

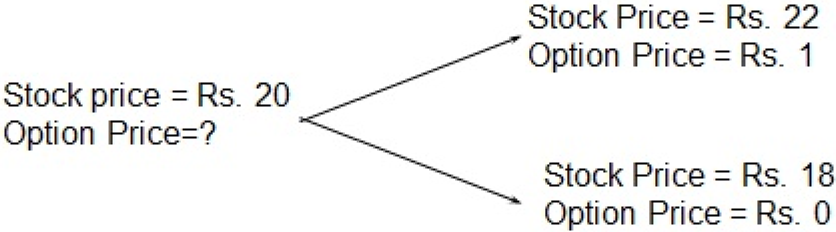
The simple random walk is the process that either moves up with a certain probability q or moves down by a fixed amount with probability $1 - q$ at each time step. We will consider a random walk which makes a move up or down by a proportion of its current state. This process will not go negative and gives a first approximation to a stock price process. The initial stock price during the period under study is $S(0)$. At each step the stock price either goes up by a factor of u (if the coin comes out heads (H), and goes down a factor of d (if the coin comes out tails (TL). $(u > 1$ and $d < 1.)$

The one step Binomial model (tree).

Consider a stock with current price $S(0)$ and an option on the stock with price (premium) f.

- Expiry time of the option $= T$,
- Suppose $S(T) = S(0)u$ with probability q and $S(T) = S(0)d$ with probability $1 - q$.
- Let the payoff from the option at time T is f_u if the stock price moves up and is f_d if the stock price moves down.
  For example, for a European call option with strike price K, the payoff at the expiry time T is $f_u = max\{(S(0)u - K), 0\}$ and $f_d = max\{(S(0)d - K), 0\}$.

A 3-month call option on the stock has a strike price of 21.



Set up a risk-less portfolio

At $t = 0$ one buys Δ shares and sells one option.
Then the value of this Portfolio at $t = 0$ is $\Delta S(0) - f$.
(Note that the cash flow is $-\Delta S(0) + f$.)
At time T,: the value of the portfolio is $\Delta S(0)u - f_u$, if the share price moves up
and is $\Delta S(0)d - f_d$, if the share price moves down.
The portfolio is thus risk less if

$$\Delta S(0)u - f_u = \Delta S(0)d - f_d,$$

i.e.

$$\Delta = \frac{f_u - f_d}{S(0)u - S(0)d}. \tag{1}$$

No arbitrage means that a risk less portfolio must earn the risk free rate of interest.
Thus we must have

$$(S(0)\Delta - f)e^{rT} = \Delta S(0)u - f_u \tag{2}$$

Putting the value Δ given in expression (1) in the above expression we get

$$f = e^{-rT}(pf_u + (1-p)f_d), \tag{3}$$

where

$$p = \frac{S(0)e^{rT} - S(0)d}{S(0)u - S(0)d} = \frac{e^{rT} - d}{u - d} = \frac{S_{now}e^{rT} - S_{down}}{S_{up} - S_{down}}. \tag{4}$$

- Note that if there is to be no arbitrage opportunity then we must have $0 < p < 1$,
  i.e. $\{p, 1-p\}$ define a probability measure (called the risk-neutral measure).
- further note that $f = e^{-rT} E_p[\text{payoff from the option at time T}]$.
- The probability p is different from and not related to the probability q (the real world measure) of the upward movement of the stock price ($\{q, 1-q\}$ is called the real world measure).

For the European call option with strike price K :

$$f = e^{-rT} E_p[max\{(S(T) - K), 0\}],$$

and for the European put option with strike price K :

$$f = e^{-rT} E_p[max\{(K - S(T)), 0\}].$$

Under the real world measure q,

$$E_q[e^{-rT}S(T)] = e^{-rT}\{qS(0)u + (1-q)S(0)d\} \neq S(0).$$

However under the risk-neutral measure p,

$$E_p[e^{-rT}S(T)] = S(0).$$

That is under p, the expected value of the share price at time T, is exactly the return on placing money equivalent to the initial stock price $S(0)$ in bonds or in a bank and earning interest at the risk free rate r. Thus if the market moves with this probability there is no reward for taking risks, both stocks and bonds give the same expected return and so investors will be neutral toward risk.

The process $\{e^{-rt}S(t), t \geq 0\}$ is a **Martingale** (w.r.t. natural filtration) under the measure p, i.e.,

$$E_p[e^{-rt}S(t)| \text{ past up to time v}] = \mathrm{e^{-rv}S(v)}.$$

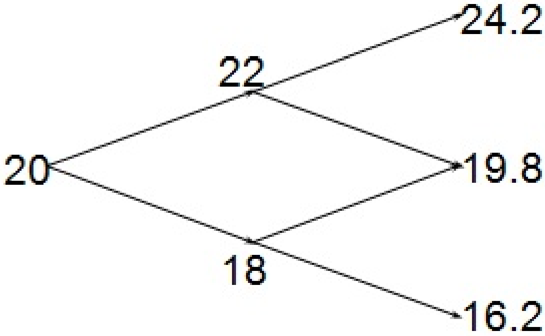
This means that the discounted stock price is a martingale. It can be shown that the existence of a probability p which gives the discounted stock price this property is equivalent to the lack of arbitrage in the market.

The fact that there is a martingale in the price process is useful for extension of this idea to continuous time models. The important point is that there should be a probability measure (a way of assigning probabilities to the paths followed by the price process) which makes the discounted price process into a martingale. The existence of such a so called equivalent martingale measure when there is no arbitrage is often called the fundamental theorem of asset pricing. The second fundamental theorem of asset pricing concerns the uniqueness of the equivalent martingale measure.

A two-step Binomial model

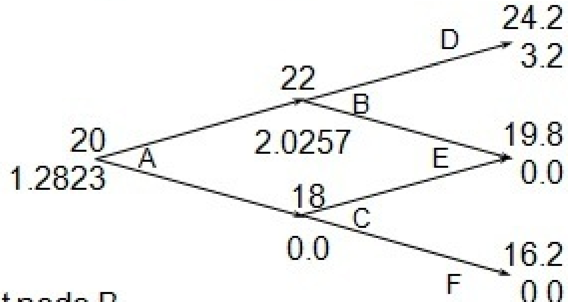
The initial stock price is $S(0)$. During each time step, it either moves up to u times its initial value or moves down to d times its initial value. Suppose the risk-free interest rate is r and the life of the option is T with the length of each time step being $T/2$. After two up movements, the value of the option is denoted by f_{uu}, after two down movements, it is denoted by f_{dd} and after one up one down movement (in any order), it is denoted by f_{du}.

Suppose $S(0) = 20$, $u = 1.1$ and $d = 0.9$.
$K = 21, \quad T = 6 \ months$ and $r = 12\% \ per \ annum$.
Then $p = \frac{e^{rT/2} - d}{u - d} = 0.6523$.



- Each time step is 3 months

European Call Option



- Value at node B
  $= e^{-0.12\times0.25}(0.6523\times3.2 + 0.3477\times0) = 2.0257$
- Value at node A
  $= e^{-0.12\times0.25}(0.6523\times2.0257 + 0.3477\times0)$
  $= 1.2823$

Repeated application of equation (3) with

$$p = \frac{e^{rT/2} - d}{u - d}$$

gives

$$f_u = e^{-rT/2}[pf_{uu} + (1-p)f_{ud}],$$
$$f_d = e^{-rT/2}[pf_{ud} + (1-p)f_{dd}]$$

and finally

$$f = e^{-rT/2}[pf_u + (1-p)f_u].$$

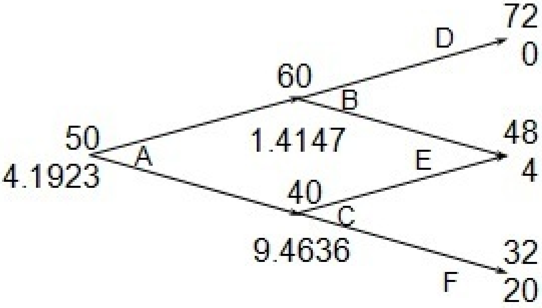
For the European options:

$$f = e^{-rT}[p^2 f_{uu} + 2p(1-p)f_{ud} + (1-p)^2 f_{dd}].$$

European Put Option

$K = 52$, time step = 1yr

$r = 5\%$ p.a.



American Put Option

$K = 52$, time step = 1yr
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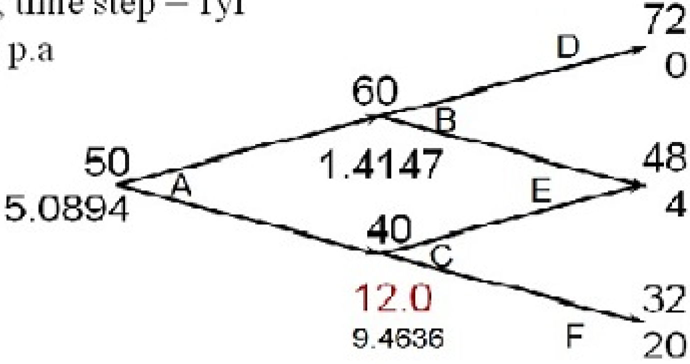
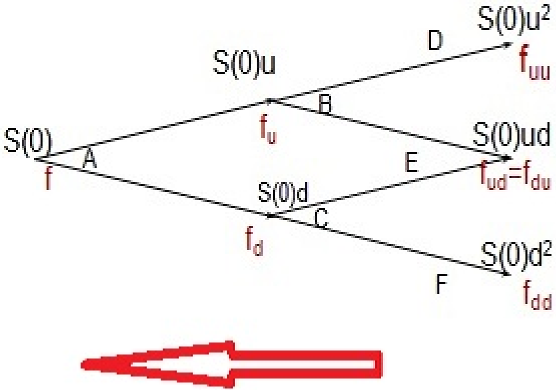


Two step Binomial Model



In practice the life T of the option is divided into a large number of small time intervals of length Δt, mostly into 30 or more time steps. In each time step a Binomial stock price movement is assumed. With 30 time steps there are 31 terminal stock prices and 2^{30} possible stock price paths. At time zero, the stock price is $S(0)$, which is known. At time Δt, there two possible stock prices, $S(0)u$ and $S(0)d$; at time $2\Delta t$, there are three possible stock prices, $S(0)u^2$, $S(0)ud$ and $S(0)d^2$. In general at time $j\Delta t$, there are $j+1$ possible stock prices, these are
$S(0)u^i d^{j-i}, \quad i = 0, 1, 2..., j.$

Options are evaluated by starting at time T, i.e., at the end of the tree and working backwards. The value of the option is known at time T. Assuming no arbitrage, the risk-neutral probability measure for each time step is

$$p = \frac{e^{r\Delta t} - d}{u - d}.$$

Note that this measure remains the same at each time step as we have assumed that in proportion of upward and downward movements remain the same for all time steps. The value of the option at each node at time $T - \Delta t$ can be calculated as the expected value under the measure p of the value of the option of time T discounted at rate r for a time period Δt. Similarly, the value of the option at each node at time $T - 2\Delta t$ can be calculated as the expected value under p of the option value of time $T - \Delta t$ discounted at rate r for a time period Δt, and so on.

In case of the American option, one has to check at each node, whether an early exercise is preferable to holding the option for a further time period Δt. The value of the option is then the maximum of the discounted expected value and the payoff obtained from the option if exercised at that time node. Working back through all the nodes, we obtain the value of the option at time zero. That is the fair price (premium) of the option so that there is no arbitrage opportunity. The unknown values involved in using the above procedure are the upward movements factor u and the downward movements factor d. In practice, these are taken to be $u = exp(\sigma\sqrt{\Delta t})$ and $d = exp(-\sigma\sqrt{\Delta t})$, where $\sigma\sqrt{\Delta t}$ is the standard deviation of the proportional change in the stock price in a short time period of length Δt and is estimated from historical data on the stock prices.

Mathematical formulation of the Binomial model for the stock price is:
The stock prices $\{S(t),\ t = 0, 1, ...\}$ form a homogeneous Markov chain with transition probabilities
$P(S_n = s_{n-1}u | S_{n-1} = s_{n-1}) = q$ and $\mathrm{P(S_n = s_{n-1}d | S_{n-1} = s_{n-1}) = (1 - q)}$. This with the initial distribution
$P(S(0) = S(0)) = 1$, we have

$$P(S_n = S(0)u^k d^{n-k} | S(0)) = \binom{n}{k} q^k (1 - q)^{n-k}.$$

The problem is to find a probability measure $\{p, 1 - p\}$ such that under this measure, the discounted stock price process is a Martingale.
In the above model, we assumed that the asset price would be one of just two specified values at time T. In reality more values are possible.

Black-Scholes-Merton model

- Fisher Black, Myron Scholes and Robert Merton made a major breakthrough in the pricing of stock options they developed the Black-Scholes (or Black-Scholes-Merton) model (1970). (*The pricing of Options and Corporate Liabilities* appearing in the Journal of Political Economy, 81 (1973) pp635–654.)
- For this work Merton and Scholes received the Nobel prize for economics in 1997.