- Continuous time; continuous variable stochastic process.
- We assume that stock prices follow Markov processes. That is, the future movements in a variable depend only on the present, and not the history of how we got to the present value.
- Martingale approach
- PDE approach

Brownian Motion (BM) or Wiener Process

Definition: A stochastic process $\mathbf{Z} = \{Z(t), t \geq 0\}$ defined on a probability space $(\Omega, \mathcal{F}, Q)$ is called a standard Brownian motion (or a standard Wiener process) if it stoc satisfies the following conditions:

1. $Z(0) = 0$, a.s. (w.p. 1).
2. $\mathbf{Z}$ has continuous sample paths w.p. 1.
3. $\mathbf{Z}$ has stationary and independent increments, i.e. for any positive integer n and any $0 = t_0 < t_1 < \cdots < t_n$, the random variables $Z(t_i) - Z(t_{i-1}), i = 1, \cdots n$ are mutually independent, and $Z(s+t) - Z(s)$ has the same distribution as Z(t) for any $s, t > 0$.
4. $Z(t)$ has the $N(0, t)$ distribution.

Bachelier (1900) seems to be the first use the Brownian motion as a model for the dynamic behavior of the Paris stock market. (Robert Brown-Botanist). Five years later Einstein (1905) developed a physical model of BM to describe motion of small particles immersed in liquid. Norbet Wiener gave the first rigorous mathematical construction.

The BM assumes positive as well as negative values with positive probability and hence the process is not a suitable model for modelling the stock price process.

The Black-Schole-Merton (BSM) model for stock price

Let $S(t)$ denote the stock price at time t and μ expected rate of return on the stock (assumed independent of t). Then in a short time interval Δt, the expected increase ΔS in S is $\mu S \Delta t$ if there is no volatility. In practice the stock price does exhibit volatility. Assume that the variability of the percentage return in a short period of time Δt does not depend on the stock price S and the time t. The model in discrete time:

$$\frac{\Delta S}{S} = \mu \Delta t + \sigma \Delta Z,$$

where Z denotes the standard BM defined on a probability space $(\Omega, \mathcal{F}, Q)$. This leads to the model:

$$dS = \mu S dt + \sigma S dZ, \qquad \mu \text{ (drift)}, \; \sigma \text{ (volatility)}$$

i.e.

$$S(t) = S(0) + \int_0^t \mu S(u) du + \sigma \int_0^t S(u) dZ(u).$$

- The BSM model: Limiting case of the random walk represented by the binomial trees as the time step $\Delta t \to 0$ and $\Delta S \to 0$ such that $\Delta S/\sqrt{\Delta t} \to \sigma($ *a constant* $> 0)$.

We assume the probability of the up and down movements to be half each and choose

$$\begin{aligned} u &= \exp\{\mu\Delta t + \sigma\sqrt{\Delta t}\} \\ d &= \exp\{\mu\Delta t - \sigma\sqrt{\Delta t}\} \end{aligned}$$

where μ and $\sigma \geq 0$ are constants and Δt is the time length of each step. We call μ the drift and σ volatility. Suppose $[0, t]$ be divided in to n time intervals each of length Δt (n $\Delta t = t$).
Let N_n denote the number of up jumps.
Then

$$S_n = S_0 u^{N_n} d^{n-N_n} = S_0 \exp\{\mu n \Delta t + \sigma\sqrt{\Delta t}\,(N_n - (n - N_n))\}$$

$$S(t) = S_n = S_0 \exp\left\{\mu t + \sigma\sqrt{t}\left(\frac{2N_n - n}{\sqrt{n}}\right)\right\}.$$

Note that N_n follows the $Bin(n, \frac{1}{2})$ distribution and by the Central Limit theorem for the i.i.d. case:

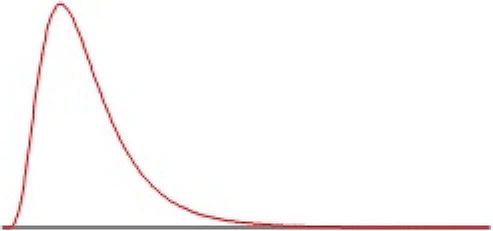
$$\frac{2N_n - n}{\sqrt{n}} = \frac{N_n - n/2}{1/2\sqrt{n}} \sim \mathcal{N}(0,1) \text{ as } \Delta\mathrm{t} \downarrow 0 \ (\mathrm{n} \to \infty).$$

Thus as $\Delta t \downarrow 0$,

$$\log S(t) \sim \mathcal{N}\left(\log S_0 + \mu t, \sigma^2 t\right),$$

that is, $S(t)$ becomes log-normally distributed.

The Lognormal Distribution



$$E(S_T) = S_0\, e^{\mu T}$$

$$\text{var}(S_T) = S_0^{\,2} e^{2\mu T}(e^{\sigma^2 T} - 1)$$

For our calculations on the discrete time binomial tree, the risk neutral probability measure

$$p = \frac{e^{r\Delta t} - d}{u - d}$$

$$\begin{aligned}
p &= \frac{\Big(\exp\{r\Delta t\} - \exp\{\mu\Delta t - \sigma\sqrt{\Delta t}\}\Big)}{\Big(\exp\{\mu\Delta t + \sigma\sqrt{\Delta t}\} - \exp\{\mu\Delta t - \sigma\sqrt{\Delta t}\}\Big)} \\
&= \frac{1 + r\Delta t - (1 + \mu\Delta t - \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t + O(\Delta t^{3/2}))}{1 + \mu\Delta t + \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t - \Big(1 + \mu\Delta t - \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t + O(} \\
&= \frac{\left(r - \mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t}}{2\sigma\sqrt{\Delta t}} = \frac{1}{2}\left(\frac{r - \mu - \frac{1}{2}\sigma^2}{\sigma}\right)\sqrt{\Delta t} + \frac{1}{2}.
\end{aligned}$$

Under the risk neutral measure p, $N_n \sim Bin(n, p)$. Now

$$\frac{2N_n - n}{\sqrt{n}} = \sqrt{4p(1-p)} \left(\frac{N_n - np}{\sqrt{np(1-p)}} + \frac{\frac{1}{2}(2p-1)\, n}{\sqrt{np(1-p)}} \right).$$

Thus as $\Delta t \downarrow 0$ $(2N_n - n)/\sqrt{n}$ is distributed approximately as

$$\mathcal{N}\left((2p-1)\sqrt{n}, 4p(1-p)\right) \quad \simeq \quad \mathcal{N}\left(\left(\frac{r - \mu - \frac{1}{2}\sigma^2}{\sigma}\right)\sqrt{t}, 1\right)$$

(Recall $S(t) = S_n = S_0 \exp\left\{\mu t + \sigma\sqrt{t}\left(\frac{2N_n - n}{\sqrt{n}}\right)\right\}$,) and thus

$$\log S(t) \sim \mathcal{N}\left(\log S_0 + \left(r - \frac{1}{2}\sigma^2\right)t, \sigma^2 t\right).$$

The effect of the risk neutral probability measure P has been to change the drift of $\log S_t$ from μ to $\left(r - \frac{1}{2}\sigma^2\right)$.

The price of the European call option then is (with $Z \sim \mathcal{N}(0,1)$ is,

$$
\begin{aligned}
& E_P\left[e^{-rT}\left(S(T)-K\right)^+\right] \\
&= E_P\left(\left(S(0)\exp\left\{-\frac{1}{2}\sigma^2 T + \sigma\sqrt{T}Z\right\} - \exp\{-rT\}K\right)^+\right) \\
&= \int_A^\infty \left(S_0\exp\left\{-\frac{1}{2}\sigma^2 T + \sigma\sqrt{T}z\right\} - \exp\{-rT\}K\right)\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{z^2}{2}\right\} dz
\end{aligned}
$$

where $A = [-rT + log(K/S(0)) + \sigma^2 T/2]/(\sigma\sqrt{T})$.

The final form of the Black-Scholes pricing formula

$$
S_0\Phi\left(\frac{\log(S_0/K) + rT + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) - e^{-rT}K\Phi\left(\frac{\log(S_0/K) + rT - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}
$$

where $\Phi(.)$ denotes the cdf of the standard normal.

- In the model
$$dS = \mu dt + \sigma dZ,$$
the expected increase in the stock price and the variability are constant in absolute terms. For example, if the expected growth rate for the stock price is Rs 5 when the stock price is Rs 25, it is also Rs 5 when the stock price is Rs 1000. Same is the case with variability.
- In the Black-Scholes-Merton(BSM) model the expected growth rate and the variability are constant when both are expressed as a proportion of the stock price $(dS/S = \mu dt + \sigma dZ.)$
- Solution: $S(t) = S(0)exp(\sigma Z(t) + (\mu - \sigma^2/2)t)$, $E_Q[S(t)] = S(0)e^{\mu t}$. (log-normal distribution). Thus μ denotes the expected rate of return.
(Under the risk neutral measure P, $E_P[S(t)] = S(0)e^{rt}$.

• The BSM model does not exhibit the 'mean reversion' effect. Some stock prices and returns tend to move back towards the mean or average or a fixed value. An increase in prices above this value is followed by a decrease and a decrease below this value is followed by an increase.

• A process with the mean reversion property: OrnsteinUhlenbeck (OU) process $\{X(t), t \geq 0\}$; a process satisfing the stochastic differential equation

$$dX(t) = \theta(\mu - X(t))\, dt + \sigma\, dZ(t),$$

where $\theta > 0$ μ and $\sigma > 0$ are parameters and $\{\ Z(t), t \geq 0\}$ a standard BM.

μ is the mean reversion parameter.

Equivalent measures:

Two measures P and Q are equivalent if they operate on the same sample space and if A is any event in the sample space, then

$$P(A) = 0 \Leftrightarrow Q(A) = 0.$$

To find a measure P equivalent to the original (real world) measure Q, such that under P the discounted process $\{e^{-rt}S_t)]\}$ is a martingale, that is, if $\mathcal{F}_t = \sigma\{S(u), 0 \leq u \leq t\}$, then

$$E_P[e^{-rt}S(t)|\mathcal{F}_u] = e^{-ru}S(u), \quad for \quad all \quad u \ \leq \ t(\leq T).$$

Cameron-Martin-Girsanov (C.M.G.)theorem:

If $\{Z_t\}$ is a Q-Brownian motion and $\{\gamma_t\}$ is an $\mathcal{F}_t$-prvisible process satisfying

$$E_Q[exp\frac{1}{2}\int_0^T \gamma_t^2 dt] < \infty,$$

then there exists a measure P such that

(i) P is equivalent to Q.

(ii) $\frac{dP}{dQ} = exp(-\int_0^T \gamma_t^2 dZ_t - \frac{1}{2}\int_0^T \gamma_t^2 dt)$ (iii) $\tilde{Z}_t = Z_t + \int_0^t \gamma_s ds$ is a P-Brownian motion.

To find a measure P under which the discounted process $\{e^{-rt}S_t)]\}$ is a martingale.
We have:

$$dS = \mu S dt + \sigma S dZ,$$

where $\{Z_t\}$ is a Q-Brownian motion. Thus

$$d(e^{-rt}S) = e^{-rt}dS + -r(e^{-rt}S)dt,$$

i.e.

$$d(e^{-rt}S) = (\mu - r)e^{-rt}S dt + \sigma e^{-rt}S dZ.$$

For the process to be a martingale the 'drift' should be zero.

In the C.M. G. theorem take $\gamma_t = (\mu - r)/\sigma$.
Then there exists a measure P such that

$$\tilde{Z}_t = Z_t + \int_0^t (\mu - r)/\sigma ds$$

is a P-Brownian motion.
Note that

$$dZ = d\tilde{Z} - (\mu - r)/\sigma dt.$$

Using the SDE for $\{e^{-rt}S\}$ we obtain:

$$d(e^{-rt}S) = (\mu - r)e^{-rt}Sdt + \sigma e^{-rt}S(d\tilde{Z} - (\mu - r)/\sigma dt),$$

i.e.

$$d(e^{-rt}S) = \sigma e^{-rt}Sd\tilde{Z},$$

which is a martingale w.r.t. the measure P.

Let X_T denote the payoff from the option at expiry time T. Then the value of the option at time t, $0 \leq t \leq T$ is :

$$f_t = e^{-r(T-t)} E_P[X_T | \mathcal{F}_t],$$

in particular

$$f = f_0 = e^{-rT} E_P[X_T].$$

Under P, the conditional distribution of $log(S(T))$ given $\mathcal{F}_t$ is

$$\sim \mathcal{N}\left(\log S_t + \left(r - \frac{1}{2}\sigma^2\right)(T-t), \sigma^2(T-t)\right).$$

For example: for the European call option with expiry time T and strike price K, the payoff $X = max\{S_T - K, 0\}$.

$$f = e^{-rT} E_P[max\{S_T - K, 0\}].$$

Now $S_t = S_0 exp(\sigma Z_t + (\mu - \sigma^2/2)t)$,

$$S_t = S_0 exp(\sigma[\tilde{Z}_t - ((\mu - r)/\sigma)t] + (\mu - \sigma^2/2)t),$$

$$S_t = S_0 exp(\sigma \tilde{Z}_t + (r - \sigma^2/2)t),$$

where $\{\tilde{Z}_t\}$ is a P-Brownian Motion.

$$e^{-rt} S_T = S_0 exp(W), \text{ where } W \text{ is } N(-(\sigma^2/2)T, \sigma^2 T).$$

Thus $f = E_P[max\{S_0 exp(W) - Ke^{-rT}, 0\}]$

$$f = \frac{1}{\sqrt{2\pi\sigma^2 T}} \int_{log\frac{K}{S_0} - rT}^{\infty} (S_0 exp(x) - Ke^{-rT}) exp\left(-\frac{(x+\sigma^2 T/2)^2}{2\sigma^2 T}\right) dx$$

i.e.

$$f = S_0 \Phi(d_1) - Ke^{-rT}\Phi(d_2),$$

where $\Phi(.)$ is the cdf of the standard normal variable,

$$d_1 = \frac{log\frac{S_0}{K} + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

and

$$d_2 = \frac{log\frac{S_0}{K} + (r - \sigma^2/2)T}{\sigma\sqrt{T}}.$$

The only unknown (but an important) parameter in the above formula is the volatility σ. This may be estimated from the historical data. The stock price is usually observed every day. Let $S_i, \ i = 0, 1, ..., n$ be a sequence of stock prices observed daily over a period of n days. and let s be the (sample) standard deviation of $ln(S_i/S_{i-1}), \ i = 1, ..., n$. Assuming that time is measured in trading days and that there are 252 trading days per year, then an estimate of σ, the volatility per annum is $\sqrt{252} \times s$.

- Implied volatility.

For a European put option: $f = e^{-rT}E_P[max\{K - S_T, 0\}]$. Thus $f = Ke^{-rT}\Phi(-d_2) - S_0\Phi(-d_1)$.

Self Financing and Replicating Portfolio

(To construct a portfolio of a risky and a risk free asset)

- $S(t)$ = price of one unit of a risky asset S at time t (no dividends)
- $B(t) = (e^{rt})$= price of one unit of a risk free security B at time t.
- $\phi(t)$ = no. of units of S held at time t and $\psi(t)$ = no. of units of B held at time t. $\phi(t)$ and $\psi(t)$ are based on the information up to (may not include) time t and hence are pre-visible w.r.t. filtartion $\{\mathcal{F}_t\}$.
- $V(t) = \phi(t)S(t) + \psi(t)B(t)$ = value of the portfolio at time t. Above prtfolio is said to be **self financing if**

$$dV(t) = \phi(t)dS(t) + \psi(t)dB(t),$$

(there is no new net investment into or out of the portfolio.)

Examples of self financing portfolios:

(i) $\phi(t) = \psi(t) = 1$ and $S(t) = Z(t)$, $(\{Z(t)\} aBM)$ $B(t) = 1$.

(ii) $S(t) = Z(t)$, $B(t) = 1$ and $\phi(t) = 2Z(t)$ $\psi(t) = -t - Z^2(t)$.

Consider a derivative with payoff (contigent claim) X_T at time T. A self financing portfolio (ϕ, ψ) is said to be a **replicating strategy** for X_T if

$$X_T = \phi(T)S(T) + \psi(T)B(T) = V(T).$$

(We want a strategy (ϕ, ψ) so that the derivative can be paid off). If there is a replicating strategy $(\phi(t), \psi(t)),\ 0 \leq t \leq T$, then under the **no arbitrage assumption** the price f(t) of the derivative at time t, $(t < T)$ is

$$f(t) = \phi(t)S(t) + \psi(t)B(t) = V(t).$$

$(f(t) > V(t)$, then sell the derivative at time t and buy the portfolio. Profit made is $f(t) - V(t) > 0$. At time T, your portfolio value is $V(T)$ which can be used to payoff the derivative. In this case profit is certain, which is an arbitrage opportunity. A reverse strategy can be used if $f(t) < V(t)$.

The law of one price: If two financial intstruments have the same payoffs (at time T), then they shoiuld have the same price at time $t < T$.

Let $\{S(t)\}$ be defined on a probability space $(\Omega, \mathcal{F}, Q)$.
The filtration $\mathcal{F}_t = \sigma(S(u), 0 \leq u \leq t)$, ($\mathcal{F}_t$ gives the information or history of the process up to time t.)
A derivative with expiry time T and payoff X_T at time T. Then X_T is $\mathcal{F}_T$-measurable (depends on events up tp time T.)

To construct a replcating strategy $(\phi(t), \psi(t))$ for X_T:

- Use the C-M-G theorem to obatin the equivalent measure P under which the discounted risky asset price process $(B(t))^{-1}S(t) = D(t)$ is a martingale w.r.t. the filtration $\{\mathcal{F}_t\}$.
- Let $E(t) = E_P[B(T)^{-1}X_T|\mathcal{F}_t]$.
  Then $\{E(t)\}$ is a martingale w.r.t. $\{\mathcal{F}_t\}$ under P.
- Since $\{E(t)\}$ and $\{D(t)\}$ are martingales under P, by **the martingale representation theorem** $\exists$ a previsible process $\{\phi(t)\}$ such that $E(t) = \int_0^t \phi(u)dD(u)$.
- Consider $\phi(t)$ units of S and $\psi(t) = E(t) - \phi(t)D(t)$ of B at time t.
- Then the value of the portfolio at time t is $V(t) = \phi(t)S(t) + \psi(t)B(t) = E(t)B(t)$.

Thus $V(T) = E(T)B(T) = B(T)E_P[(B(T))^{-1}X_T|\mathcal{F}_T]$,
but X_T is $\mathcal{F}_T$-measurable, thus

$$V(T) = B(T)(B(T))^{-1}X_T = X_T,$$

i.e. (ϕ, ψ) is a replicating strategy for X_T.
Further under the no arbitrage assumption

$$f(t) = V(t) = B(t)E_P[B(T)^{-1}X|\mathcal{F}_t],$$

$$f(t) = e^{-r(T-t)}E_p[X_T|\mathcal{F}_t]$$

and

$$f(0) = e^{-rT}E_P[X_T|\mathcal{F}_0] = e^{-rT}E_P[X_T]$$

The pay off from a derivative will be a function of $S(t), 0 \leq t \leq T$, say $h(S())$.
Since the (conditional) distribution of $S(t)$ w.r.t. P is known, Monte Carlo techniques can be used to compute the expectaion,

$$e^{-r(T-t)} E_P[h(S())|\mathcal{F}_t].$$

Another approach: Black-Scholes-Merton partial differential equation for the price of a derivative.

Ito's Lemma

Assume that the process S satisfies the following stochastic differential equation (SDE)

$$dS = a(S,t)dt + b(S,t)dZ(t),$$

where $a(S,t)$ and $b(S,t)$ are adapted processes. Let f be a twice continuous differentiable function then

$$df(S,t) = \left(\frac{\partial f}{\partial t} + a(S,t)\frac{\partial f}{\partial S} + \frac{1}{2}b^2(S,t)\frac{\partial^2 f}{\partial^2 S}\right)dt + b(S,t)\frac{\partial f}{\partial S}dZ$$

S: the stock price; f:the price of the derivative

Example

Suppose $S(t)$ satisfies

$$dS = \mu \; Sdt + \sigma \; SdZ(t),$$

then $G = ln(S)$ satisfies the SDE

$$dG = (\mu - \sigma^2/2)dt + \sigma dZ.$$

($\frac{\partial G}{\partial t} = 0$, $\frac{\partial G}{\partial S} = \frac{1}{S}$ and $\frac{\partial^2 G}{\partial^2 S} = -\frac{1}{S^2}$,
$a(S,t) = \mu S$ and $b(S,t) = \sigma \; S$.)

Discrete versions assuming Black-Scholes model:

$$\Delta S = \mu S \Delta t + \sigma S \Delta Z,$$

$$\Delta f = \left(\frac{\partial f}{\partial t} + \mu S \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S \frac{\partial^2 f}{\partial^2 S} \right) \Delta t + \sigma S \frac{\partial f}{\partial S} \Delta Z$$

Set up a risk free portfolio:
Buy $\frac{\partial f}{\partial S}$ shares and sell one option.
Then the value V of the portfolio is:

$$V = -f + \frac{\partial f}{\partial S} S.$$

The change in the value of this portfolio in the small time interval Δt is

$$\Delta V = -\Delta f + \frac{\partial f}{\partial S} \Delta S$$

$$\Delta V = \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\sigma^2 S \frac{\partial^2 f}{\partial^2 S}\right) \Delta t.$$

The above equation does not involve the random term ΔZ and hence must be risk free during time Δt. No arbitrage assumption implies that the portfolio should earn the same rate of return r as a risk-free security during that time period.

Thus

$$\Delta V = rV\Delta t, \quad (\approx V(e^{r\Delta t} - 1))$$

$$-\frac{\partial f}{\partial t} - \frac{1}{2}\sigma^2 S \frac{\partial^2 f}{\partial^2 S} = r\left(-f + \frac{\partial f}{\partial S} S\right)$$

.

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S \frac{\partial^2 f}{\partial^2 S} = r\ f.$$

- Solve given the boundary conditions.
- The boundary condition for the European call option is $f(T) = max\{S(T) - K, 0\}$. The solution is

$$f(t) = S(t)\Phi(d_1(t)) - Ke^{-r(T-t)}\Phi(d_2(t)),$$

where $\Phi(.)$ is the cdf of the standard normal variable,

$$d_1(t) = \frac{log\frac{S(t)}{K} + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}$$

and

$$d_2(t) = d_1(t) - \sigma\sqrt{T - t}.$$

- The boundary condition for the European put option is $f(T) = max\{K - S(T), 0\}$. The solution is

$$f(t) = Ke^{-r(T-t)}\Phi(-d_2(t)) - S(t)\Phi(-d_1(t)),$$

(put-call parity)

- The boundary condition for the (long) forward contract with the delivery price K is

$$f(T) = S(T) - F$$

The solution is

$$f(t) = S(t) - Ke^{-r(T-t)}.$$

At the beginning of the life of the contract, the delivery price $K = F$ the forward price $(F = S(0)E^{rT})$, and the value of the contract $f = 0$. As time passes the delivery price remains the same (because it is part of the definition of the contract) but the forward price changes.

• The price of any derivative on an asset whose price satisfies the BSM model should satisfy the above equation under the no artbitrage assumption.
Examples:

- $f(t) = e^S$ does not satisfy the BSM differential equation and hence can not be the price of a derivative dependent on the stock price under the no 'arbitrage' assumption.
- $f(t) = \frac{e^{(\sigma^2 - r)(T-t)}}{S}$ satisfies the equation and under the no 'arbitrage' assumption is the price of a derivative with payoff $1/S(T)$ at time T.

Derivatives on interest rates

Term Structure of the rate.

- Let $P(t, T)$ denote the price of zero coupon bond at time t, $0 \leq t \leq T$, where T is the time of maturity of the bond. (It is assumed that the bond pays Rs 1 at the time of maturity.)
- Yield to Maturity $\{Y(t, T)\}$:

$$P(t, T) = e^{-Y(t,T)(T-t)}.$$

- Short rate $r(t)$: $r(t) = \lim_{T \to t} Y(t, T)$. (interest rate in small interval of time Δt).
- $\frac{\partial P(t,T)}{\partial t} = r(t)P(t, T)$.
- $P(t, T) = e^{-\int_t^T r(s)ds}$.

Models for the short rate

(Equlibrium models in the risk neutral world)

Rendleman and Bartter:

$$dr = ardt + \sigma r dZ.$$

Vasicek:

$$dr = a(b - r)dt + \sigma dZ.$$

Cox, Ingersoll and Ross (CIR):

$$dr = a(b - r)dt + \sigma\sqrt{r}dZ.$$

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