- Continuous time; continuous variable stochastic process.
- We assume that stock prices follow Markov processes. That is, the future movements in a variable depend only on the present, and not the history of how we got to the present value.
- Martingale approach
- PDE approach

The BSM model:

$$dS = \mu S dt + \sigma S dZ,$$

where $\{Z_t\}$ is a Q-Brownian motion.
Using the Girsanov-Martin-Cameron theorem we obtain a probability measure P equivqlent to Q such that

$$\tilde{Z}_t = Z_t + \frac{\mu - r}{\sigma} t$$

is a P-Brownian motion.
Further,

$$d(e^{-rt} S) = \sigma e^{-rt} S d\tilde{Z},$$

which is a martingale w.r.t. the measure P.

Self Financing and Replicating Portfolio

(To construct a portfolio of a risky and a risk free asset)

- $S(t)$ = price of one unit of a risky asset S at time t (no dividends)
- $B(t) = (e^{rt})$= price of one unit of a risk free security B at time t.
- $\phi(t)$ = no. of units of S held at time t and $\psi(t)$ = no. of units of B held at time t. $\phi(t)$ and $\psi(t)$ are based on the information up to (may not include) time t and hence are pre-visible w.r.t. filtartion $\{\mathcal{F}_t\}$.
- $V(t) = \phi(t)S(t) + \psi(t)B(t)$ = value of the portfolio at time t. Above prtfolio is said to be **self financing if**

$$dV(t) = \phi(t)dS(t) + \psi(t)dB(t),$$

(there is no new net investment into or out of the portfolio.)

Examples of self financing portfolios:

(i) $\phi(t) = \psi(t) = 1$ and $S(t) = Z(t)$, $(\{Z(t)\} aBM)$, $B(t) = 1$.

(ii) $S(t) = Z(t),\ B(t) = 1$ and $\phi(t) = 2Z(t)\ \ \psi(t) = -t - Z^2(t)$.

Consider a derivative with payoff (contigent claim) X_T at time T. A self financing portfolio (ϕ, ψ) is said to be a **replicating strategy** for X_T if

$$X_T = \phi(T)S(T) + \psi(T)B(T) = V(T).$$

(We want a strategy (ϕ, ψ) so that the derivative can be paid off). If there is a replicating strategy $(\phi(t), \psi(t)),\ 0 \leq t \leq T$, then under the **no arbitrage assumption** the price f(t) of the derivative at time t, $(t < T)$ is

$$f(t) = \phi(t)S(t) + \psi(t)B(t) = V(t).$$

$(f(t) > V(t)$, then sell the derivative at time t and buy the portfolio. Profit made is $f(t) - V(t) > 0$. At time T, your portfolio value is $V(T)$ which can be used to payoff the derivative. In this case profit is certain, which is an arbitrage opportunity. A reverse strategy can be used if $f(t) < V(t)$.)

Let $\{S(t)\}$ be defined on a probability space $(\Omega, \mathcal{F}, Q)$.
The filtration $\mathcal{F}_t = \sigma(S(u), 0 \leq u \leq t)$, ($C_t$ gives the information or history of the process up to time t.)
Consider a derivative with expiry time T and payoff X_T at time T.
Then X_T is $\mathcal{F}_T$-measurable (depends on events up to time T.)
Thus $E[X_T|\mathcal{F}_T] = X_T$.

To construct a replcating strategy $(\phi(t), \psi(t))$ for X_T

:

- Use the C-M-G theorem to obatin the equivalent measure P under which the discounted risky asset price process $(B(t))^{-1}S(t) = D(t)$ is a martingale w.r.t. the filtration $\{\mathcal{F}_t\}$.
- Let $E(t) = E_P[B(T)^{-1}X_T|\mathcal{F}_t]$.
  Then $\{E(t)\}$ is a martingale w.r.t. $\{\mathcal{F}_t\}$ under P.
- Since $\{E(t)\}$ and $\{D(t)\}$ are martingales under P, by **the martingale representation theorem** $\exists$ a previsible process $\{\phi(t)\}$ such that $E(t) = \int_0^t \phi(u)dD(u)$.
- Consider $\phi(t)$ units of S and $\psi(t) = E(t) - \phi(t)D(t)$ of B at time t.
- Then the value of the portfolio at time t is $V(t) = \phi(t)S(t) + \psi(t)B(t) = E(t)B(t)$.

Thus $V(T) = E(T)B(T) = B(T)E_P[(B(T))^{-1}X_T|\mathcal{F}_T]$,
but X_T is $\mathcal{F}_T$-measurable, thus

$$V(T) = B(T)(B(T))^{-1}X_T = X_T,$$

i.e. (ϕ, ψ) is a replicating strategy for X_T.
Further under the no arbitrage assumption

$$f(t) = V(t) = B(t)E_P[B(T)^{-1}X|\mathcal{F}_t],$$

$$f(t) = e^{-r(T-t)}E_p[X_T|\mathcal{F}_t]$$

and

$$f(0) = e^{-rT}E_P[X_T|\mathcal{F}_0] = e^{-rT}E_P[X_T]$$

The pay off from a derivative will be a function of $S(t), 0 \leq t \leq T$, say $h(S())$.
Since the (conditional) distribution of $S(t)$ w.r.t. P is known, Monte Carlo techniques can be used to compute the expectaion,

$$e^{-r(T-t)} E_P[h(S())|\mathcal{F}_t].$$

Under P, the conditional distribution of $log(S(T)$ given $\mathcal{F}_t$ is

$$\sim \mathcal{N}\left(\log S_t + \left(r - \frac{1}{2}\sigma^2\right)(T-t), \sigma^2(T-t)\right).$$

Another approach: Black-Scholes-Merton partial differential equation for the price of a derivative.

Ito's Lemma

Assume that the process S satisfies the following stochastic differential equation (SDE)

$$dS = a(S,t)dt + b(S,t)dZ(t),$$

where $a(S,t)$ and $b(S,t)$ are adapted processes. Let f be a twice continuous differentiable function then

$$df(S,t) = \left(\frac{\partial f}{\partial t} + a(S,t)\frac{\partial f}{\partial S} + \frac{1}{2}b^2(S,t)\frac{\partial^2 f}{\partial^2 S}\right)dt + b(S,t)\frac{\partial f}{\partial S}dZ$$

S: the stock price; f:the price of the derivative

Example

Suppose $S(t)$ satisfies

$$dS = \mu \ Sdt + \sigma \ SdZ(t),$$

then $G = ln(S)$ satisfies the SDE

$$dG = (\mu - \sigma^2/2)dt + \sigma dZ.$$

($\frac{\partial G}{\partial t} = 0$, $\frac{\partial G}{\partial S} = \frac{1}{S}$ and $\frac{\partial^2 G}{\partial^2 S} = -\frac{1}{S^2}$,
$a(S,t) = \mu S$ and $b(S,t) = \sigma \ S$.)

Discrete versions assuming Black-Scholes model:

$$\Delta S = \mu S \Delta t + \sigma S \Delta Z,$$

$$\Delta f = \left(\frac{\partial f}{\partial t} + \mu S \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S \frac{\partial^2 f}{\partial^2 S} \right) \Delta t + \sigma S \frac{\partial f}{\partial S} \Delta Z$$

Set up a risk free portfolio:
Buy $\frac{\partial f}{\partial S}$ shares and sell one option.
Then the value V of the portfolio is:

$$V = -f + \frac{\partial f}{\partial S} S.$$

The change in the value of this portfolio in the small time interval Δt is

$$\Delta V = -\Delta f + \frac{\partial f}{\partial S} \Delta S$$

$$\Delta V = \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\sigma^2 S \frac{\partial^2 f}{\partial^2 S}\right)\Delta t.$$

The above equation does not involve the random term ΔZ and hence must be risk free during time Δt. No arbitrage assumption implies that the portfolio should earn the same rate of return r as a risk-free security during that time period.

Thus

$$\Delta V = rV\Delta t, \quad (\approx V(e^{r\Delta t} - 1))$$

$$-\frac{\partial f}{\partial t} - \frac{1}{2}\sigma^2 S \frac{\partial^2 f}{\partial^2 S} = r\left(-f + \frac{\partial f}{\partial S} S\right)$$

.

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S \frac{\partial^2 f}{\partial^2 S} = r\ f.$$

- Solve given the boundary conditions.
- The boundary condition for the European call option is $f(T) = max\{S(T) - K, 0\}$. The solution is

$$f(t) = S(t)\Phi(d_1(t)) - Ke^{-r(T-t)}\Phi(d_2(t)),$$

where $\Phi(.)$ is the cdf of the standard normal variable,

$$d_1(t) = \frac{log\frac{S(t)}{K} + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}$$

and

$$d_2(t) = d_1(t) - \sigma\sqrt{T - t}.$$

- The boundary condition for the European put option is $f(T) = max\{K - S(T), 0\}$. The solution is

$$f(t) = Ke^{-r(T-t)}\Phi(-d_2(t)) - S(t)\Phi(-d_1(t)),$$

(put-call parity)

- The boundary condition for the (long) forward contract with the delivery price K is

$$f(T) = S(T) - F$$

The solution is

$$f(t) = S(t) - Ke^{-r(T-t)}.$$

At the beginning of the life of the contract, the delivery price $K = F$ the forward price ($F = S(0)e^{rT}$) , and the value of the contract $f = 0$. As time passes the delivery price remains the same (because it is part of the definition of the contract) but the forward price changes.

• If the asset price satisfies the BSM model then the price of a derivative on it should satisfy the above equation under the no artbitrage assumption.

Examples:

- $f(t) = e^S$ does not satisfy the BSM differential equation and hence can not be the price of a derivative dependent on the stock price under the no 'arbitrage' assumption.
- $f(t) = \frac{e^{(\sigma^2 - r)(T-t)}}{S}$ satisfies the equation and is the price of a derivative with payoff $1/S(T)$ at time T, under the no 'arbitrage' assumption.

Mixed Jump Diffusion model:

$$dS = (\mu - \lambda k)Sdt + \sigma SdZ + SdY,$$

where $\{Y_t, t \geq 0\}$ is a jump (compound Poisson) process,
k is the expected size of the jumps,
$\lambda \, \Delta t$ is the probability that a jump occurs in the next interval of length Δt.
(Girsanov theorem for jump processes; existence of multiple risk neutral measures.)
Jumps have a big effect on the implied volatility of short term options. They have a much smaller effect on the implied volatility of long term options

Constant elasticity variance (CEV) model:

$$dS = (r - q)Sdt + \sigma S^{\alpha} dZ,$$

$\alpha = 1$ is the BSM model,
$\alpha > 1$: volatility rises as stock price rises,
$\alpha < 1$: volatility decreases as stock price rises,
(If $0 < \alpha < 1$, $S(t)$ can be zero with positive probability.)
Unique risk neutral measure.

Stochastic volatility models:

$$dS = (r - q)Sdt + \sqrt{V}SdZ$$

$$dV = a(V_L - V)dt + \sigma V^{\alpha} dZ_V,$$

V and S are uncorrelated. Price of an European option is the Black-Scholes price integrated over the distribution of the average variance.
When V and S are negatively correlated we obtain a downward sloping volatility skew, which seems to be similar to that observed in the market for equities. When V and S are positively correlated the skew is upward sloping.

The European and American call and put options are termed as plain vanilla options. They have well defined properties and are traded actively. They prices are qouted by exchanges on a regular basis.

Some exotic options: In the-over -the-counter derivatives markets non standard (exotic) options are created by the financial engineers.

- Package options: a portfolio consisting of standard European calls and puts; for example, a long call and a short put.
- Nonstandard American options: early exercise is restricted is restricted to certain dates. Strike price changes over the life. Forward start options: option starts at a future point T_1. (Most common in employee stock option plans)

- Compound options: option to buy or sell an option (call on call, put on call, ...)
- Chooser options: option starts at 0 matures at T_2 at $T_1,\ 0 < T_1 < T_2$ the buyer choses whether it is put or call.
- Barrier options: option comes in to existence only if the stock price hits a barrier (in option)// Option dies if the stock price hits a barrier (out option).
- Binary options: Cash or nothing. Asset or nothing.
- Lookback options: Look back call pays $S(T) - min(S(t)$ at time T. Lookback put pays $maxS(t) - S(T)$ at time T.

- Shout options: buyer can shout once during the life of the option.

- Asian options: depends on the average price of the asset.
  call: $max(S_{ave} - K)$

- Options to exchange one asset for another

- Options involving several assets.

Derivatives on interest rates

Term Structure of the rate.

- Let $P(t, T)$ denote the price of zero coupon bond at time t, $0 \leq t \leq T$, where T is the time of maturity of the bond. (It is assumed that the bond pays Rs 1 at the time of maturity.)

- Yield to Maturity $\{Y(t, T)\}$:

$$P(t, T) = e^{-Y(t,T)(T-t)}.$$

- Short rate $r(t)$: $r(t) = \lim_{T \to t} Y(t, T)$. (interest rate in small interval of time Δt).

- $\frac{\partial P(t,T)}{\partial t} = r(t)P(t, T)$.

- $P(t, T) = e^{-\int_t^T r(s)ds}$.

Models for the short rate

(Equlibrium models in the risk neutral world)

Rendleman and Bartter:

$$dr = ardt + \sigma rdZ.$$

Vasicek:

$$dr = a(b - r)dt + \sigma dZ.$$

Cox, Ingersoll and Ross (CIR):

$$dr = a(b - r)dt + \sigma \sqrt{r} dZ.$$

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