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Approaches to Damage Models and related results in Applied Probability

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Abstract

Approaches based on non-negative matrices or, in particular, on specialized versions of de Finetti's theorem, have led to some results of relevance to damage models, including those on Integrated Cauchy Functional Equation and Extended Spitzer Integral Representation Theorem. Now we revisit these results shedding further light on some of their aspects; in the process of doing this, we observe, amongst other things, that the latter of the two results referred to here has a link with the Weyl integral from fractional calculus.

Keywords: Rao's damage model, Lau-Rao-Shanhag theorems, Spitzer integral representation theorem, Rao-Rubin-Shanbhag theorems, Non-negative matrices, Fractional calculus, Weyl integral, Deny's theorem, de Finetti's theorem, Branching processes, Markov processes.

1 Introduction

The notion of damage models was introduced, giving some motivation and relevant supporting material, by Rao (1963). This has generated considerable interest amongst researchers specializing in characteristic properties of stochastic models and related integral equations.

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In mathematical terms, a damage model defined by Rao may be viewed as a random vector $(\mathbf{X}, \mathbf{Y})$ with non-negative integer-valued components $\mathbf{X}, \mathbf{Y}$ such that $\mathbf{Y} \leq \mathbf{X}$ almost surely with $0 < P\{\mathbf{Y} = \mathbf{X}\} \leq 1$; in the context of a damage model, the conditional distribution of $\mathbf{Y}$ given $\mathbf{X}$, i.e. any chosen version of it, is usually referred to as the survival distribution of the model, and the following as the Rao-Rubin (RR(0)) condition:

$$P\{\mathbf{Y} = y\} = P\{\mathbf{Y} = y | \mathbf{X} - \mathbf{Y} = 0\}, \qquad y = 0, 1, \ldots \quad . \tag{1.1}$$

The multivariate versions of the terminologies and certain of their variations have also appeared in the literature; see e.g. Chapter 7 of Rao and Shanbhag (1994) or Rao and Shanbhag (2004).

Amongst various important results that one comes across on damage models, there are those that are due to Rao and Rubin (1964) and Shanbhag (1977); in particular, from what is observed in the latter article, it follows that under certain conditions, the problem of identifying the solution to (1.1) reduces to that of solving a discrete version of an integral equation, studied, terming it as the Integrated Cauchy Functional Equation (ICFE), in Lau and Rao (1982). Also, it may be noted here that more general versions of ICFE have essentially been studied in Chapter 3 of Rao and Shanbhag (1994) and in other places such as, Shanbhag (1991).

The main result of Shanbhag (1977) subsumes several specialized results, including that of Rao and Rubin (1964); Chapter 7 of Rao and Shanbhag (1994) highlights various important results appearing in the literature on damage models. Included in these, besides the result of Shanbhag and certain of its extensions and variations, is an extended version of Spitzer's integral representation theorem relative to stationary measures of certain discrete branching processes, with a link to damage models. More recently, Rao and Shanbhag (2004) have shown that this latter result holds even when one of the moment assumptions in it, i.e. in the notation used in the literature, that m^* is finite, is dropped.

Rao and Shanbhag [(1994, Section 4.4), (1998), (2004)] and Rao et al. (2002) have shown explicitly or otherwise that there exist approaches to damage models based on non-negative matrices or exchangeability, involving amongst other things, certain special cases of de Finneti's theorem. Chapters 2 and 3 of Rao and Shanbhag (1994) provide us with further information on some of the research material met in the references such as Alzaid et al. (1987b), Rao and Shanbhag (1991) and Shanbhag (1991), involving ideas based on exchangeability or, in particular, de Finetti's theorem, to solve certain versions of Choquet-Deny (1960) and Deny (1961) equations, or their variations. Also, from the cited literature, it is now evident that we have proofs based on versions of de Finetti's theorem or ICFE for certain potential theoretic results such as Hausdorff's theorem on completely monotone sequences, and Bernstein's and Bochner's theorems on completely monotone functions (and hence also, their versions on absolutely monotone functions).

In the paper, we make some new observations on integral equations, of relevance to damage models, involving partially aspects of non-negative matrices or exchangeability, and show, in particular, that the Weyl integral met in fractional calculus has a link with the extended Spitzer integral representation theorem referred to above. We also highlight in this paper some of the major implications of these findings.

2 Generalized Discrete ICFE with application to damage models

Before dealing with the main findings on the generalized discrete ICFE in this section, we shall briefly revisit an application of such equations to damage models, met essentially in Rao and Shanbhag (2004, pp. 67-68):

Let S be a countable Abelian semigroup with zero element, equipped with discrete topology, and v and w be non-negative real-valued functions defined on S, with v satisfying additionally that $v(0) > 0$. Note that there are cases of $(S, v.w)$, in which

$$v(x) = \sum_{y \in S} v(x+y)w(y), \qquad x \in S. \tag{2.1}$$

Given (S, w), possibly, meeting some additional conditions, the problem of identifying the (class of) functions v, for which (2.1) is valid, may be viewed as that of solving a version of discrete ICFE. Suppose $S^* \subset S$ (not depending on w) is such that, in the case of $supp(w)(= \{x : w(x) > 0\}) \supset S^*$, (2.1) is met if and only if (iff, for short) there exists a family $\{e(x, .) : x \in S\}$ of non-negative random variables defined on a probability space, meeting the requirements that $e(x, .)e(y, .) = e(x + y, .), x, y \in S$, $\sum_{x \in S} e(x, .)w(x) = 1$ and $E(e(x, .)) = \frac{v(x)}{v(0)}, x \in S$. (There is obviously no loss of generality if we consider in (2.1), in place of v, its normalised version with $v(0) = 1$.)

Assume now that $a : S \to (0, \infty)$ and $b : S \to [0, \infty)$ are such that $b(0) > 0$ and there exists $c : S \to (0, \infty)$ as the convolution of a and b, and Y and Z are random elements defined on a probablility space, with values in S, such that

$$P\{Y = y, Z = z\} = g(y+z)\frac{a(y)b(z)}{c(y+z)}, \qquad y, z \in S,$$

where $\{g(x) : x \in S\}$ is a probability distribution. If $supp(b) \supset S^*$, then it easily follows that

$$P\{Y = y\} = P\{Y = y | Z = 0\}, \qquad y \in S,$$

iff $\frac{g(x)}{c(x)} \propto E(e^*(x,.)), x \in S$, where $\{e^*(x,.)\}$ meets the requirements of $\{e(x,.)\}$, referred to above, but, with, for some $\gamma > 0$, $\gamma.b$ appearing in place of w.(Note that by assumption, $c(x) \geq a(x)b(0) > 0$ for each $x \in S$ and c exists at least in the cases with a, b bounded and $\sum_{x \in S} a(x) < \infty$ or $\sum_{x \in S} b(x) < \infty$.)

Let $S^*(w)$ be the smallest subsemigroup of S containing $\{0\} \cup supp(w)$. Then, if $S = S^*(w)$, the existence of S^*, with stated property, is implied by what is observed as an application of a result on non-negative matrices in Rao and Shanbhag (1994, pp. 98-99) or by an argument based on a version of deFinetti's theorem produced in Rao and Shanbhag (1998, Section 2) to obtain, essentially, a criterion for the validity of (2.1).

The information that we have gathered above tells us, in particular, that the following theorem holds; for some results of relevance to this theorem, see, also, Ressel (1985) and Rao et al. (2002).

Theorem 2.1: *Let S be as defined earlier. Also, let $v : S \to [0, \infty)$ with, $v(0) = 1$, $w : S \to [0, \infty)$ and $S^*(w)$ be as defined above. Then*

$$v(x) = \sum_{y \in S} v(x+y)w(y), \qquad x \in S^*(w), \tag{2.2}$$

iff there exists a family $\{e(x,.) : x \in S^*(w)\}$ of non-negative random variables defined on a probability space, satisfying $e(x,.)e(y,.) = e(x+y,.), x, y \in S^*(w)$, $\sum_{x \in S^*(w)} e(x,.)w(x) = 1$ and $E(e(x,.)) = v(x), x \in S^*(w)$.

(We use here the notation e again for simplicity, but in a different context.)

Remark 2.1: *Theorem 2.2 of Rao and Shanbhag (2004) proved via a simple version of deFinetti's theorem, is indeed a corollary to Theorem 2.1 appearing above in which $S = S^*(w) = (\{0, 1, 2...\})^k$ (k being a positive integer). The cited article obtains (the multivariate versions of) Hausdorff's theorem on completely monotone sequences and Rao-Rubin-Shanbhag theorems on damage models, as obvious corollaries to this result; see, also, Rao and Shanbhag (1994, pp. 166-167) for some relevant findings on damage models.*

Remark 2.2: *We can shed further light on the link between Hausdorff's theorem and the specialized version of Theorem 2.1, referred to in the previous remark. In the case of* $S = S^*(w) = (\{0, 1, 2...\})^k$*, w of Theorem 2.1 is such that $w(x) > 0$ for each x of length 1, enabling us to define (v^*, w^*), where (in obvious notation) for each $x(= (x_1, x_2, .., x_k))$, $v^*(x) = v(x) \prod_{r=1}^k w_r^{x_r}$ and $w^*(x) = w(x) \prod_{r=1}^k w_r^{-x_r}$ with $w_r = w(\delta_{r1}, ..., \delta_{rk})$, $r = 1, .., k$, involving the standard notation of Kronecker delta. Suppose in the case of $S = S^*(w) = (\{0, 1, 2...\})^k$, (v, w) satisfies (2.2), then, in the notation of Rao and Shanbhag (1994, p.75), for each $(n_1, n_2, .., n_k) \in (\{0, 1, 2...\})^k$, one can observe inductively (with respect to $(m_1, m_2, .., m_k)$) that (2.2) holds, with $((-1)^{m_1+m_2+...+m_k} \Delta_1^{m_1} ... \Delta_k^{m_k} v^*, w^*)$ in place of (v, w) and meeting the condition that both of its components are non-negative (but, not necessarily with the first one as non-vanishing or normalized). In view of this, since we have now also that $\sum_{x \in S} v^*(x) w^*(x) = \sum_{x \in S} \sum_{y \in S} v^*(x + y) w^*(x) w^*(y) = 1$, (in conjunction with Fubini's theorem,) Hausdorff's theorem implies easily the "only if" part of the specialized version of Theorem 2.1; the "if" part of this latter result being obvious, we are thus effectively led to a different proof for a specialized version in question.*

Remark 2.3: *The last remark met above tells us that the multivariate generalization of Shanbhag's (1977) lemma (i.e. Theorem 2.2 of Rao and Shanbhag (2004)) can also be proved via an approach based on Hausdorff's theorem, with the proof simplifying considerably in the univariate case. Also, in view of what we now know, especially from Rao and Shanbhag (2004, pp.65-66), the following version of Bochner's theorem can be dealt with as a corollary not only to Hausdorff's theorem, but, also to the extended version of Shanbhag's lemma; clearly, Bernstein's theorem corresponding to completely monotone functions on $(0, \infty)$ is an immediate corollary to this.*

"**Theorem 2.2** *A function $f : ((0, \infty))^k \to [0, \infty)$ is completely monotone iff, for some non-negative measure ν concentrated on $([0, \infty))^k$ and determined uniquely by f,*

$$f(x) = \int_{([0,\infty))^k} exp\{- < x, y >\} d\nu(y), \qquad x \in ((0, \infty))^k,$$

with $< ., . >$ denoting the usual inner product."

Remark 2.4: *Taking a clue from the proof of Corollary 2.1 in Rao and Shanbhag (2004), we note that any completely monotone function f on $(0, \infty)^k$ satisfies, for $n_1, n_2, .., n_k \in \{0, 1, ...\}$ and $x_1, x_2, .., x_k \in (0, \infty)$,*

$$H(n_1, n_2, .., n_k, x_1, x_2, .., x_k)$$

$$= \frac{1}{k} \sum_{r=1}^{k} \sum_{m=1}^{\infty} 2^{-m} \{ \int_0^{1/m} H(n_1 + \delta_{r1}, .., n_k + \delta_{rk}, x_1 + \delta_{r1} y, ..., x_k + \delta_{rk} y) dy$$

$$+ H(n_1, n_2, .., n_k, x_1 + \tfrac{\delta_{r1}}{m}, x_2 + \tfrac{\delta_{r2}}{m}, .., x_k + \tfrac{\delta_{rk}}{m}) \},$$

where $\delta_{rr'}$ is Kronecker delta and H is so that for $n'_1, n'_2, .., n'_k \in \{0, 1, ...\}$ and $x'_1, x'_2, .., x'_k \in (0, \infty)$,

$$H(n'_1, n'_2, .., n'_k, x'_1, x'_2, .., x'_k) = (-\frac{\partial}{\partial x'_1})^{n'_1} ... (-\frac{\partial}{\partial x'_k})^{n'_k} f(x'_1, x'_2, .., x'_k).$$

Hence, if f is a completely monotone function on $(0, \infty)^k$, then H corresponding to it meets the requirements of h of Corollary 3.4.5 of Rao and Shanbhag (1994), with $S^*(\mu) = S$. Taking into account this information, in conjunction with the uniqueness theorem for Laplace-Stieltjes transforms relative to non-negative measures, it can easily be verified that the theorem for completely monotone functions on $(0, \infty)^k$, met in Remark 2.3 above, and its counterpart for absolutely monotone functions on $(-\infty, 0)^k$, hold. Also, that we have now alternative arguments for proving Theorems 3.5.1 and 3.5.2 of Rao and Shanbhag (1994), with obvious advantages, especially, in the latter case, is clear.

Remark 2.5: *If g is an absolutely monotone function on $\prod_{r=1}^{k}(0, a_r)$, then $g(e^{\theta_1}, e^{\theta_2}, ..., e^{\theta_k})$, $(\theta_1, ..., \theta_k) \in \prod_{r=1}^{k}(-\infty, ln\ a_r)$, turns out to be absolutely monotone and implies, in view of the relevant result on absolutely monotone functions, referred to in the previous remark, that*

$$g(z_1, .., z_k) = \int_{([0,\infty))^k} (\prod_{r=1}^{k} z_r^{x_r}) d\nu(\underline{x}), \qquad (z_1, .., z_k) \in \prod_{r=1}^{k}(0, a_r),$$

(in obvious notation) with ν as a non-negative measure concentrated on $([0, \infty))^k$. It may now be noted, in particular, that if g has the integral representation with respect to ν, then, unless ν is concentrated on $(\{0, 1, ...\})^k$, one has the existence of positive integers r, r', so that $\lim_{z_r \to 0} (\frac{\partial}{\partial z_r})^{r'} g(z_1, .., z_k) = \infty$, contradicting the assertion that g be absolutely monotone on $\prod_{r=1}^{k}(0, a_r)$. This, in turn, implies that g meets the condition that it is absolutely monotone on $\prod_{r=1}^{k}(0, a_r)$ iff, for some $\nu : (\{0, 1, ...\})^k \to [0, \infty)$,

$$g(z_1,..,z_k) = \sum_{x_1=0}^{\infty} ... \sum_{x_k=0}^{\infty} (\prod_{r=1}^{k} z_r^{x_r}) \nu(\{(x_1,...,x_k)\})\ (z_1,..,z_k) \in \prod_{r=1}^{k}(0,a_r).$$

The univariate version of the conclusion reached here appears essentially in Feller (1966, p.222), although the approach used in this case is different.

Taking a clue from Rao and Shanbhag (1994, 4.4.3, iv)), we now give the following interesting corollary to Theorem 2.1.

Corollary 2.1: *Let S, v, w and $S^*(w)$ be as in Theorem 2.1. Also, let, for each $x \in (S^*(w))^c$, there be $u_x : S \to [0,\infty)$ such that $x + supp(u_x) \subset S^*(w)$ and*

$$v(x+y) = \sum_{z \in S} v(x+y+z)u_x(z), \qquad y \in S. \tag{2.3}$$

Then the assertion of Theorem 2.1 holds with S in place of $S^*(w)$.

Proof: The "if" part of the assertion follows easily in view of Fubini's theorem. To establish the "only if" part of the assertion, assume then that the relevant version of (2.2) holds. In view of the validity of the "only if" part of Theorem 2.1, we can assume the existence of the probability space and $\{e(x,.) : x \in S^*(w)\}$ as in the theorem, and extend, for each ω (in obvious notation), $e(x,\omega), x \in S^*(w)$, to $e(x,\omega)$, $x \in S$, such that, for each $x \in (S^*(w))^c$,

$$e(x,\omega) = \begin{cases} \sum_{y \in supp(u_x)} e(x+y,\omega)u_x(y) & \text{if this is finite} \\ 0 & \text{otherwise,} \end{cases}$$

u_x being as mentioned in the statement of the corollary. Applying Fubini's theorem, it can be seen that if (2.3) is valid,

$$v(x_1+x_2+x_3+x_4) = E(\prod_{r=1}^{4} e(x_r,.)), \qquad x_1,x_2,x_3,x_4 \in S. \tag{2.4}$$

Taking x in place of x_1 and letting $x_2 = x_3 = x_4 = 0$, since $v(0) = 1$, from (2.4), it is easily seen that $E(e(x,.)) = v(x)$, $x \in S$. Also, if we now choose $x_1,x_2,x_3,x_4 \in S$ such that $x_1+x_2 = x_3+x_4$, (2.4) implies that

$$E((e(x_1,.)e(x_2,.) - e(x_3,.)e(x_4,.))^2) = 0 \tag{2.5}$$

and, then, letting $x_3 = x_1 + x_2$ and $x_4 = 0$ in (2.5), that $e(x_1,.)e(x_2,.) = e(x_1+x_2,.)$ a.s. Since S is countable, we have hence the existence of a nullset

N such that for each $\omega \in N^c$, $e(x_1, \omega)e(x_2, \omega) = e(x_1 + x_2, \omega)$, $x_1, x_2 \in S$. Redefining then the restriction of e to $S \times N$ so as to have for all $\omega \in N$, $e(x, \omega) = e(x, \omega_0)$, $x \in S$, with $\omega_0 \in N^c$, we obtain a version of e meeting the requirements of the "only if" part of the assertion. Hence, it follows that the result referred to holds and we have the corollary. □

Remark 2.6: *If there exists another family, viz,* $\{u'_x : x \in (S^*(w))^c\}$ *meeting the requirement of* $\{u_x : x \in (S^*(w))^c\}$ *of Corollary 2.1, then, denoting, for convenience, the extension of e to S met in the proof of the corollary, but, with* $\{u'_x\}$ *in place of* $\{u_x\}$, *by* e', *it can be seen that (2.4) in the proof of the corollary remains valid if, for some* r, $e'(x_r, .)$ *appear in place of* $e(x_r, .)$. *This, in turn, implies then that* $E((e(x, .) - e'(x, .))^2) = 0$, $x \in S$, *and, hence, that, for each* $x \in S$, $e(x, .) = e'(x, .)$ *a.s.*

Remark 2.7: *Our goal in this article, or, in particular, in this section, is not to give an exhaustive review of the potential theoretic results on moments or Laplace-Steiltjes transforms, but to shed, in a simple way, as much light as possible on the link between the multivariate version of Hausdorff's theorem on completely monotone sequences and a version of ICFE on* $(\{0, 1, ..\})^k$. *In the process of doing this, we, also, make an effort to highlight the approaches based on the discrete ICFE for verifying that Bernstein's theorem for completely monotone functions on* $(0, \infty)$, *and its multivariate extension appearing in Bochner(1960, pp. 86-87), hold; this has obvious implications to the versions of the two theorems referred to here, concerning absolutely monotone functions. For an account of the early research on Bernstein's theorem, involving especially, a proof of the theorem based on Hausdorff's theorem, we refer the reader to Widder(1946, Chapter IV, pp. 160-164). It may also be worth pointing out in this place that the versions of Bernstein's and Bochner's theorems, for certain absolutely monotone functions have proved useful in some damage model studies, see, e.g., Rao and Rubin (1964) and Talwalker (1970).*

3 Extended Spitzer integral representation theorem and its relation to Weyl integral

Let $m \in (0, 1)$ and f be the probability generating function (pgf, for short) of a non-negative integer valued random variable with mean m. Also, let, for each $c \in (0, 1]$, $U_c : [0, 1) \to [0, \infty)$, with $U_c(0) = 0$ and $U_c(f(0)) = 1$, and for each $c \in (0, 1)$, $G_c : [0, 1) \to [0, \infty)$ with $G_c(0) = 1$, denote the restrictions to $[0, 1)$ of the generating functions (gf's, for short) of some sequences $\{u_c(n) :$

$n = 0, 1, ...\}$ and $\{g_c(n) : n = 0, 1, ...\}$ with $u_c(n), g_c(n) \in [0, \infty)$, $n = 0, 1, ...$, respectively. Then, we have the following Theorem 3.1 and Corollary 3.1, respectively, on the two sets of functions that we have introduced.

Some specialized versions of the results referred to, especially, of Theorem 3.1, or, the results related to these have been dealt with by Spitzer (1967), Athreya and Ney (1972, Chapter II), Alzaid et al. (1987a) and; amongst others, by Rao et al. (2002). From the literature, it is now evident that the functional equations considered in these results are of relevance to the identifiability problem concerning stationary measures of certain discrete branching processes, or, concerning discrete probability distributions for which a modified version of the Rao-Rubin condition is met.

That Theorem 3.1 holds is implied, in view of a certain uniqueness property, relative to Laplace-Stieltjes transforms, essentially, by what is observed in Rao and Shanbhag (2004, pp. 69-70), and that Corollary 3.1 follows from the theorem can be easily seen.

Theorem 3.1: *Given f (and hence $m \in (0, 1)$) and $c \in (0, 1]$, any U_c, of the form referred to , satisfies*

$$cU_c(f(s)) = c + U_c(s), \qquad s \in [0, 1), \tag{3.1}$$

iff, for some probability measure ν on $[0, 1)$,

$$U_c(s) = \int_{[0,1)} U_c(s, t)(U_c(f(0), t))^{-1} d\nu(t), \qquad s \in [0, 1), \tag{3.2}$$

where

$$U_c(s, t) = \sum_{n=-\infty}^{\infty} [exp\{(B(s)-1)m^{n-t}\} - exp\{-m^{n-t}\}]c^n, \quad s, t \in [0, 1), \tag{3.3}$$

and B is the unique pgf among those vanishing at $s = 0$, for which,

$$B(f(s)) = m\ B(s) + 1 - m, \qquad s \in [0, 1]. \tag{3.4}$$

Moreover, (with f, c fixed,) (3.2) specifies a one-to-one correspondence between the class of functions U_c satisfying (3.1) and the class of probability measures ν on $[0, 1)$. (For an interpretation of B in branching processes, see Athreya and Ney (1972, p 17).)

Proof: The first part of the theorem is a reorganized version of Theorem 3.1 of Rao and Shanbhag (2004) and hence that it is valid is obvious. Also, now it can be easily seen that, for each probability measure ν on $[0, 1)$, there exists a function U_c satisfying (3.2) and hence (3.1). Conversely, for each U_c satisfying (3.1), (3.2) is met with ν as a probability measure on $[0,1)$. Recall now that, if μ is a non-negative measure on $[0,\infty)$ satisfying the condition that, for some open interval I (in $\mathbb{R}$), there exists $\Phi : I \to [0,\infty)$ with $\Phi(\theta) = \int_{[0,\infty)} exp\{-x\theta\} d\mu(x), \theta \in I$, then Φ determines μ. In view of this, it follows that, under the assumptions implied, for U_c satisfying (3.2), $-(B^{-1})'(1-\theta)U_c'(B^{-1}(1-\theta)), \theta \in (0,1)$, (in obvious notation) determines $\int_{[0,x)} m^{-t}(U_c(f(0),t))^{-1} d\nu(t)$, $x \in [0,1)$, and hence ν. This, in turn, confirms that the assertion concerning a one-to-one correspondence between the classes of U_c and ν, respectively, appearing in the second part of the theorem is valid.(See Remark 3.4 for further relevant information.) □

Corollary 3.1: *Given f (and hence m) and $c \in (0,1)$, any G_c of the form considered satisfies*

$$c\, G_c(f(s)) = G_c(s), \qquad s \in [0,1), \tag{3.5}$$

iff, for some probability measure ν on $[0,1)$,

$$G_c(s) = \int_{[0,1)} G_c(s,t)(G_c(0,t))^{-1} d\nu(t), \qquad s \in [0,1), \tag{3.6}$$

where

$$G_c(s,t) = \sum_{n=-\infty}^{\infty} c^n exp\{(B(s)-1)m^{n-t}\}, \qquad s,t \in [0,1), \tag{3.7}$$

with B as in Theorem 3.1. Also, under the assumptions, (3.6) determines a one-to-one correspondence between the class of functions G_c satisfying (3.5) and the class of probability measures ν on $[0,1)$.

Proof: The corollary follows from Theorem 3.1 since there exists a one-to-one correspondence between the class of U_c satisfying (3.1) and G_c satisfying (3.5), determined by

$$U_c(s) = c(1-c)^{-1}(G_c(s)-1), \qquad s \in [0,1). \qquad \square \tag{3.8}$$

Remark 3.1: *Corresponding to each f, we have B (with $B(0) = 0$ and,) $B(f(0)) = 1 - m$ and hence, for each $t \in [0,1)$,*

$$U_c(f(0),t) = \begin{cases} 1 & \text{if } c = 1 \\ (c^{-1}-1)G_c(0,t) & \text{if } c < 1, \end{cases}$$

where $U_c(s,t)$ and $G_c(s,t)$ are as in (3.3) and (3.7), respectively. Hence, as a corollary to Theorem 3.1, it follows that, for each U_c for which (3.1) is met, $U_c(B^{-1}(s)), s \in [0,1)$, satisfies the specialized version of (3.1) with $1-m+ms$ in place of $f(s)$, and, hence, also that of (3.2) with s in place of $B(s)$. That an analogous statement is valid for G_c is now obvious. Also, we may stress here that we don't insist upon restricting to the cases of B with finite mean in either of Theorem 3.1 and Corollary 3.1; the following simple example, including that of Harris referred to in Athreya and Ney (1972, p.72) for $c=1$, illustrates this point in a simple way:

Example: Let f and B, the corresponding pgf satisfying (3.4), be so that the latter does not have any restriction on its mean (i.e. on the mean of the corresponding distribution). Then, U_c so that, for each $s \in [0,1)$,

$$U_c(s) = \begin{cases} (ln\ (1-B(s)))(ln\ m)^{-1} & \text{if } c = 1 \\ c(1-c)^{-1}(G_c(s)-1) & \text{if } c < 1, \end{cases}$$

with $G_c(s) = (1-B(s))^{-(ln\ c)/(ln\ m)}$, satisfies (3.1). Also, G_c involved in this example is a function for which (3.5) is met.

Remark 3.2: *Under appropriate assumptions, the classes of U_c satisfying (3.1) and G_c satisfying (3.5) are convex possessing extreme points given respectively by (3.2) and (3.6), with ν degenerate. The extreme points in the two cases are given, for each $t \in [0,1)$, respectively by $U_c(s,t)(U_c(f(0),t))^{-1}, s \in [0,1)$ and $G_c(s,t)(G_c(0,t))^{-1}$, $s \in [0,1)$, and we shall refer to these as extremal functions. (Incidentally, the key observations appearing in this remark are by-products of Theorem 3.1 and Corollary 3.1, respectively.)*

Remark 3.3: *If, for $c_1, c_2 \in (0,1)$, G_{1,c_1} and G_{2,c_2} denote versions of G_c satisfying (3.5) with $c = c_1$ and $c = c_2$, respectively, then there exists a version G_{3,c_1c_2} of G_c satisfying (3.5) with $c = c_1c_2$, such that*

$$G_{3,c_1c_2}(s) = \prod_{r=1}^{2} G_{r,c_r}(s), \qquad s \in [0,1). \tag{3.9}$$

Corollary 3.1 implies that the product on the right hand side of (3.9) has a representation of the form of (3.6) with $c = c_1c_2$ and ν having infinite support points. Consequently, it follows that (3.9) can't be valid with G_{3,c_1c_2} as extremal or a mixture of finitely many extremals.

Remark 3.4: *Incidentally, in the argument used to prove the second part of Theorem 3.1, there is no loss of generality if one assumes that $B(s) =$*

s. In this case, the argument simplifies and also relates to the one-to-one correspondence between the extension of U_c' (in obvious notation) to $(-\infty, 1)$ as an absolutely monotone function, and the measure in its representation given by Bernstein's theorem. (For an application of Bernstein's theorem to the extension of U_c', see Alzaid et al (1987a, p.1213)). Given U_c, to identify ν in (3.2), it is sufficient to determine the restriction to $[1, m^{-1})$ of the measure μ (in the notation implied in the proof of Theorem 3.1) relative to $-U_c'(1-\theta), \theta \in (0,1)$, or of the measure relative to the representation for the extension (to $(-\infty, 1)$) of U_c', given by Bernstein's theorem.(We adopt in this remark as well as in what follows, the concept of differentiability for any real-valued function on [0,1) in obvious way, classifying the function as differentiable if it is right differentiable at 0 and its restriction to (0,1) is differentiable, and use standard notation to denote derivatives in this case.)

Remark 3.5: *With obvious changes to a relevant argument in Alzaid et al. (1987a, p.1212), it can be seen, as a simple corollary to Yaglom's theorem, referred to in Athreya and Ney (1972, p.18), that for each U_c satisfying (3.1) and $s \in [0,1)$ (with f_n as the n^{th} iterate of f, met in branching processes) $\lim_{n\to\infty} c^n(U_c(f_n(0)+(1-f_n(0))s) - U_c(f_n(0))$ exists and equals $U_c(B^{-1}(s))$. This, in turn, implies, in view of the extended continuity theorem of Feller (1966, p.433), that $U_c(B^{-1}(s)), s \in [0,1)$, meets the requirements of U_c, satisfying (3.1), with $f(s) = 1 - m + ms$. Since Remark 2 of Alzaid et al (1987a), which assumes implicitly (as observed in Remark 3.8 of Rao et al. (2003)) that $f'(0) > 0$, applies to the specialized version of f, it follows that $U_c(B^{-1}(s)), s \in [0,1)$, satisfies (3.2) with $B(s) = s$. This obviously leads us to an alternative approach for checking that the "only if" part of the first assertion of Theorem 3.1 holds.*

Remark 3.6: *Incidentally, the approach based on Yaglom's theorem, met in the previous remark, tells us, in view of (3.4), that, for each $s \in [0,1]$, $\lim_{n\to\infty} m^{-n}(1 - B(f_n(0) + (1-f_n(0))s))$ exists and equals $1 - s$.*

Remark 3.7: *The authors have noticed some blemishes in Rao and Shanbhag (2004). In particular, in the cited paper, in (1.2) on page 62,"$P\{X = y\}$" should have appeared as "$P\{Y = y\}$", in (i) in the second paragraph on page 63, "(1.1)" should have appeared as "(1.2)", and in Remark 3.4, appearing on pages 71 and 72, a correction is needed, such as that line 3 onwards in it, the places of "$U^*_{(c)}$" and "$U^*_{(1)}$" be interchanged, with the specific example of $U^*_{(1)}$ (on page 72) replaced by that of extremal $U^*_{(c)}$.*

We shall henceforth restrict to the case with $f'(0) = m$, i.e. to the case in which the support of the distribution relative to f is $\{0,1\}$, and understand by U_c and G_c, for respective c, as those satisfying (3.1) and (3.5) respectively, with f as mentioned here. Also, following essentially the approach of Alzaid et al. (1987a, p.1212-1213), given in different notation, to extend U_c, we can now extend G_c to $(-\infty, 1)$, and denoting the relevant extension of G_c by $\widehat{G_c}$, observe that

$$c\, \widehat{G}_c(1 - m + ms) = \widehat{G}_c(s), \qquad s \in (-\infty, 1). \tag{3.10}$$

In the notation, we can then, give the following theorem, linking Theorem 3.1 and Corollary 3.1 to the Weyl integral; for the details of the Weyl integral, see Oldman and Spanier (1974, p.53).

Theorem 3.2: *The assertions that appear below are valid:*

(i) Given $c \in (0, 1]$ and $n \in \{1, 2, ...\}$, in each case, $U_c^{(n)}$, the function relative to the n^{th} derivative of U_c, is proportional to some G_{cm^n}, and, conversely, each G_{cm^n} is proportional to $U_c^{(n)}$ relative to some U_c and, if $c < 1$, also to $G_c^{(n)}$ (in obvious notation) relative to some G_c.

(ii) If $c, c' \in (0, 1)$ and n is a positive integer such that $n > (ln(\frac{c'}{c}))/(ln(m))$, then, given $\widehat{G_c}$, there exists a $\widehat{G_{c'}}$, satisfying (in obvious notation)

$$\widehat{G}_{c'}(s) \propto \int_0^\infty \widehat{G}_c^{(n)}(s - y) y^{n-(ln(\frac{c'}{c}))/(ln(m))-1} dy, \qquad s \in (-\infty, 1), \tag{3.11}$$

involving a version of the Weyl integral, and, conversely, given $\widehat{G_{c'}}$, there exists a $\widehat{G_c}$ satisfying the same relation.

Proof: The first part of (i) is obvious, while, its second part follows inductively if we prove it just for $n = 1$. By Fubini's theorem, we see (in obvious notation) that, for each $s \in [0, 1)$,

$$\int_0^s G_{cm}(y)dy = \sum_{n=1}^{\infty} (g_{cm}(n-1)) \frac{s^n}{n} = cm \int_0^s G_{cm}(1 - m + my)dy$$

$$= c \sum_{n=1}^{\infty} (g_{cm}(n-1))(\frac{(1 - m + ms)^n}{n} - \frac{(1-m)^n}{n}).$$

Then defining $\{u_c(n) : n = 0, 1, ..\}$ such that $u_c(0) = 0$ and $\{u_c(n) : n = 0, 1, ..\}$ is an appropriate constant multiple of $\{n^{-1} g_{cm}(n-1) : n = 1, 2, ..\}$, it follows that, by (3.8), $U_c^{'}$ in this case is proportional to G_{cm}; if $c < 1$, we have also the existence G_c with G'_c proportional to G_{cm}. This completes the proof of (i).

That (ii) follows from Corollary 3.1 is easy to verify. However, we shall now use a somewhat different approach to show that the assertion is valid. To do this, we first note that, since $\widehat{G}_c$ satisfies (3.10), for each non-negative integer k, the function $\overline{\Phi}_{k,c} : \mathbb{R} \to (0, \infty)$ such that, for each $x \in \mathbb{R}$,

$$\overline{\Phi}_{k,c}(x) = exp\{(k + ln\ c/ln\ m)x\}\widehat{G}_c^{(k)}(1 - e^x),$$

is periodic with period $ln\ (m^{-1})$ and continuous. Consequently, it follows that the integral appearing in (3.11) exists as a positive real function on $(-\infty, 1)$ satisfying (3.10) with c' in place of c. Since $\widehat{G}_c$ is absolutely monotone on $(-\infty, 1)$, we have, for each non-negative integer k,

$$\widehat{G}_c^{(n+k)}(s - y) = \int_{-\infty}^{s} \widehat{G}_c^{(n+k+1)}(x - y)dx, \qquad y \in (0, \infty), s \in (-\infty, 1),$$

and, hence, by Fubini's theorem, the function defined by the integral in (3.11), is seen to be proportional to a version of $\widehat{G}_{c'}$; this proves the first part of (ii). To see the validity of the converse statement, let $\widehat{G}_{c'}$ be arbitrary. We have then, applying Fubini's theorem and the first part of (ii),

$$\widehat{G}_{c'}(s) = \int_0^{\infty} \widehat{G}_{c'}^{(n+n')}(s - y)y^{n+n'-1}dy$$

$$\propto \int_0^{\infty} \widehat{G}_{c'}^{(n+n')}(s - y)(\int_0^{y} z^{n-\alpha-1}(y - z)^{n'+\alpha-1}dz)dy$$

$$= \int_0^{\infty} (\int_0^{\infty} \widehat{G}_{c'}^{(n+n')}(s - z - x)x^{n'+\alpha-1}dx)z^{n-\alpha-1}dz$$

$$\propto \int_0^{\infty} \widehat{G}_c^{(n)}(s - z)z^{n-\alpha-1}dz,$$

where $\alpha = (ln\ (\frac{c'}{c}))/(ln\ (m))$, n' is the integer replacing n in (3.11) when the places of c and c' are interchanged, and $\widehat{G_c}$ is that determined by $\widehat{G}_{c'}$ in view of the relevant version of (3.11). This completes the proof of (ii). □

Remark 3.8: *If we take $c = m$ in (3.11), the second part of Theorem 3.2 implies essentially that, in the present case of f, the representation for U_c , $c < 1$, can be obtained from that for U_1.*

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