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Random Central Limit Theorem for Associated Random Variables and the Order of Approximation

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Abstract : We consider a random number N_n and strictly stationary associated random variables X_k and form random partial sums $S_{N_n} = \sum_{j=1}^{N_n} X_j$ under the assumption that N_n is independent of the sequence $\{X_r, r \geq 1\}$ for every $n \geq 1$. Under certain conditions on the random variables X_k and N_n, we obtain the limit distribution of the sequence S_{N_n} after suitable normalization. We also obtain an estimate of the order of approximation.

1 Introduction

A set of random variables $\{X_1, X_2, \ldots, X_k\}$ is said to be associated if for each pair of coordinatewise nondecreasing functions $f, g : R^k \to R$,

$$Cov(f(X_1, X_2, \ldots, X_k), g(X_1, X_2, \ldots, X_k)) \geq 0$$

whenever the covariance exists. A sequence $\{X_n, n \geq 1\}$ of random variables is said to be associated if, for every $n \in N$, the family $X_1, X_2, \ldots, X_n$ is associated.

For examples of sequences of associated random variables and their properties, see Prakasa Rao and Dewan (2001) and Prakasa Rao (2012). Let the sequence $\{X_n, n \geq 1\}$ be a strictly stationary sequence of square integrable associated random variables. Central limit theorem (CLT) for the sequence $\{X_n, n \geq 1\}$ was proved by Newman (1980) (cf. Bulinski and Shaskin (2007), Prakasa Rao (2012), Oliveira (2012)).

Let $\{X_n, n \geq 1\}$ be a stationary associated sequence of square integrable random variables. Let $EX_n = \mu, V(X_1) = \sigma_1^2, c_j = Cov(X_1, X_{1+j}), S_n = \sum_{j=1}^{n} X_j$ and $\sigma^2 = \sigma_1^2 + 2\sum_{j=1}^{\infty} c_j$.

We assume one or more of the following assumptions on the sequence of covariances $\{c_j, j \geq 1\}$ in the sequel:
(A1)$\sum_{j=1}^{\infty} c_j < \infty$;
(A2)$\sum_{j=1}^{\infty} j c_j < \infty$;
(A3)$\sum_{j=n}^{\infty} c_j < Cn^{-\theta}$ for all n and some positive constants C and θ independent of n.

Note that each of the conditions (A2) and (A3) implies (A1). Wood (1983) called the condition (A1) as "finite susceptibility" condition. We now recall the central limit theorem of Newman (1980) for a sequence of stationary associated random variables.

Theorem 1.1: *Let a stationary sequence of square integrable associated random variables* $\{X_n, n \geq 1\}$ *satisfy the condition (A1) and* $\sigma^2 > 0$. *Then*

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \stackrel{D}{\to} Z_1 \sim N(0,1) \quad \text{as} \quad n \to \infty$$

where $N(0,1)$ *denotes the standard normal distribution.*

The following theorem is a consequence of the result in Wood (1983).

Theorem 1.2: (Corollary 4.14, Oliveira (2012)) *Let a stationary sequence of square integrable associated random variables* $\{X_n, n \geq 1\}$ *satisfy the condition (A3) and* $\sigma^2 > 0$. *Further suppose that* $E|X_1|^3 < \infty$. *Then there exists a positive constant* C *such that*

$$\sup_{x \in R} |P(S_n - n\mu \leq \sqrt{n}\sigma x) - \Phi(x)| < Cn^{-\frac{1}{5}} \tag{1. 1}$$

where $\Phi(x)$ *denotes the distribution function of the standard normal random variable.*

Barbour and Xia (2006) discussed normal approximation for random sums. Gnedenko and Korolev (1996) investigated random summation and limit theorems for random sums. Improved Berry-Esseen type bounds for partial sums of sequences of associated random variables were derived in Birkel (1988), Bulinski (1995) and Dewan and Prakasa Rao (1997, 2005) under alternate sets of conditions. Central limit theorems for random sums of independent random variables and order of approximations are investigated for some dependent sequences such as martingales, mixing sequences and m-dependent random sequences (cf. Islak (2013), Landers and Rogge (1976, 1988), Prakasa Rao (1969, 1974, 1975), Shang (2012), Sreehari (1968, 1975), Sunklodas (2014), Tomko (1967)). Prakasa Rao and Sreehari (1982, 2015) obtained the order of approximation in the random central limit theorem for m-dependent random variables. Applications of these limit theorems is essential in the study of statistical inference problems by methods of sequential analysis where the sample size at the termination of sampling is random. As far as we are aware, there are no results on central limit theorems for random sums of associated random variables. We prove such a result now for the first time and also obtain an estimate on the order of approximation. We prove some lemmas in Section 2 and the random central limit theorem for associated sequences of random variables is proved in Section 3. An estimate of the remainder term in the random central limit theorem is obtained in Section 4.

2 Assumptions and Lemmas

Consider a sequence $\{X_n\}$ of strictly stationary square integrable random variables with covariances $c_j \geq 0$ satisfying the assumption (A1).
Note that

(2. 1) $$V(S_n) = n\sigma_1^2 + 2n\sum_{j=1}^{n-1} c_j - 2\sum_{j=1}^{n-1} jc_j$$

and, by the Kronecker lemma, the condition (A1) implies that

(2. 2) $$\frac{V(S_n)}{n} \to \sigma^2 < \infty$$

as $n \to \infty$.

Let the sequence $\{N_n\}$ be a sequence of nonnegative integer valued random variables such that, for each n, the random variable N_n is independent of the random variables $\{X_r, r \geq 1\}$ and is such that N_n, properly normalized, converges in distribution to a random variable Z_2. We assume that the random variables N_n are such that

(A4)
$$\frac{EN_n}{n} \to \nu > 0, \frac{V(N_n)}{n} \to \tau^2 < \infty$$

as $n \to \infty$, $\nu\sigma^2 + \mu^2\tau^2 > 0$ and that, for large n,

(2. 3) $$\sup_{x \in R} \left| P(N_n - EN_n \leq x\sqrt{V(N_n)}) \ - \ G(x) \right| < \epsilon_n$$

where G is a continuous distribution function satisfying the condition that there exists a positive constant C such that

(2. 4) $$\sup_{x \in R} |G(x+y) - G(x)| < Cy, \, y > 0$$

and the sequence $\epsilon_n \to 0$ as $n \to \infty$.

In view of the inequality in (2.3) and the assumptions (A4), it follows that

$$\frac{N_n - E(N_n)}{V(N_n)} \xrightarrow{P} 0 \text{ as } n \to \infty. \tag{2. 5}$$

We now compute $V(S_{N_n})$.

Lemma 2.1: *Let $P(N_n = k) = p_{n,k}$. Then, under the condition (A1),*

$$V(S_{N_n}) = E(N_n)\,\sigma^2 + V(N_n)\,\mu^2 - 2\sum_{j=1}^{\infty} jc_j P(N_n > j) - 2\sum_{j=1}^{\infty} c_j \left(\sum_{k=0}^{j} kp_{n,k}\right). \tag{2. 6}$$

Proof : Proceeding as in the proof of Lemma 2.1 in Prakasa Rao and Sreehari (2015), we have

$$\begin{aligned} V(S_{N_n}) &= EN_n\sigma_1^2 + V(N_n)\mu^2 \\ &\quad +2\sum_{k=0}^{\infty}\left[p_{n,k}\left\{k\sum_{j=1}^{\infty} c_j I(j<k) - \sum_{j=1}^{\infty} jc_j I(j<k)\right\}\right]. \end{aligned} \tag{2. 7}$$

Now

$$\sum_{k=0}^{\infty} p_{n,k} k(\sum_{j=1}^{\infty} c_j I(j<k)) = \sum_{j=1}^{\infty} c_j(\sum_{k=j+1}^{\infty} kp_{n,k}) = \sum_{j=1}^{\infty} c_j[EN_n - \sum_{k=0}^{j} kp_{n,k}]. \tag{2. 8}$$

Furthermore

$$\sum_{k=0}^{\infty} p_{n,k}(\sum_{j=1}^{\infty} jc_j I(j<k)) = \sum_{j=1}^{\infty} jc_j P(N_n > j). \tag{2. 9}$$

From (2.7) - (2.9) above, we get (2.6).

Remarks: Observe that, if the assumption (A2) holds, then

$$\sum_{j=1}^{\infty} jc_j P(N_n > j) \le \sum_{j=1}^{\infty} jc_j < \infty$$

and

$$\sum_{j=1}^{\infty} c_j(\sum_{k=0}^{j} kp_{n,k}) \le \sum_{j=1}^{\infty} jc_j(\sum_{k=0}^{j} p_{n,k}) \le \sum_{j=1}^{\infty} jc_j < \infty,$$

so that, by the assumptions concerning EN_n and $V(N_n)$, it follows that

$$\frac{V(S_{N_n})}{n} \to \nu\sigma^2 + \mu^2\tau^2 \tag{2. 10}$$

as $n \to \infty$. In the next section, we will prove that

$$\frac{S_{N_n} - E(S_{N_n})}{\sqrt{V(S_{N_n})}} \xrightarrow{D} Z^* \tag{2. 11}$$

as $n \to \infty$ where Z^* is a linear function of independent random variables Z_1 and Z_2 and obtain the rate of convergence in the limit theorem stated above in Section 4. It will be noted that, if Z_2 is also a standard normal random variable, then Z^* is also standard normal.

We now state a lemma which is of independent interest.

Lemma 2.2: *Let U_n and U be random variables with the distribution function of U Lipschtzian with constant $\alpha > 0$ and V be a random variable independent of U_n and U with $E|V| < \infty$. Let $g : R \to R$. Then, for any constant c and positive δ and for all $z \in R$,*

$$\begin{aligned} &|P(U_n + Vg(U_n) \le z) - P(U + cV \le z)| \\ &\le \sup_{x \in R} |P(U_n \le x) - P(U \le x)| + P(|g(U_n) - c| > \delta) + \alpha\delta\, E|V| \end{aligned} \tag{2. 12}$$

and

$$\begin{aligned} &|P(U_n + Vg(U_n) \le z) - P(U_n + cV \le z)| \\ &\le 2 \sup_{x \in R} |P(U_n \le x) - P(U \le x)| + P(|g(U_n) - c| > \delta) + 2\alpha\delta\, E|V|. \end{aligned} \tag{2. 13}$$

Remarks: The first inequality (2.12) given above is Lemma 2.3 in Prakasa Rao and Sreehari (2015). The proof of the second inequality is similar to that of the first inequality.

3 Random Central Limit Theorem (RCLT)

Before we state and prove the main result of this section, we need to introduce some notation.

Let

$$d_K(U, V) = \sup_{x \in R} |P(U \le x) - P(V \le x)|$$

be the Kolmogorov distance between the distribution functions of U and V. Define

$$T_n = \frac{S_{N_n} - E(S_{N_n})}{\sqrt{V(S_{N_n})}} = \frac{S_{N_n} - \mu N_n}{\sqrt{V(S_{N_n})}} + \frac{(N_n - EN_n)\mu}{\sqrt{V(S_{N_n})}},$$

and

$$T_n(Z_1) = \sqrt{\frac{N_n}{V(S_{N_n})}}\sigma Z_1 + \frac{(N_n - EN_n)\mu}{\sqrt{V(S_{N_n})}}$$

where Z_1 is a standard normal random variable and is independent of the sequence $\{N_n\}$. Furthermore, define

$$T'_n(Z_1) = \sqrt{\frac{\nu}{\nu\sigma^2 + \mu^2\tau^2}}\sigma Z_1 + \frac{(N_n - EN_n)\mu}{\sqrt{V(S_{N_n})}}$$

and

$$T(Z_1, Z_2) = \frac{\mu\tau}{\sqrt{\nu\sigma^2 + \mu^2\tau^2}}\left[\frac{\sigma\sqrt{\nu}}{\mu\tau}Z_1 + Z_2\right]$$

where Z_2 follows the distribution function G given at (2.3) and is independent of Z_1. We will prove that the random variable $T(Z_1, Z_2)$ is the limit random variable Z^* in (2.11).

In the following discussion, the letter C with or without subscript will denote a positive constant.

Theorem 3.1: *Let the sequence $\{X_n\}$ be a stationary associated sequence of square integrable random variables satisfying the condition (A2). Let the sequence $\{N_n\}$ be a sequence of nonnegative integer valued random variables such that, for each n, the random variable N_n is independent of the sequence $\{X_k\}$ satisfying the assumption (A4). Then*

$$d_K(T_n, T(Z_1, Z_2)) \to 0 \quad as \quad n \to \infty$$

where the random variables Z_1 and Z_2 are as defined earlier.

Proof : Set $D_n = \{|N_n - n\nu| \le n\nu/2\}$. Then

(3. 1)
$$\sup_{x\in R}|P(T_n \le x) - P(T_n(Z_1) \le x)|$$
$$\le \sum_{n\nu/2\le k\le 3n\nu/2} p_{n,k}\sup_{x\in R}\left|P\left(\frac{S_k - \mu k}{\sigma\sqrt{k}} \le x(n,k)\right) - P(Z_1 \le x(n,k))\right| + P(D'_n)$$

where

$$x(n,k) = \frac{x\sqrt{V(S_{N_n})} - \mu(k - EN_n)}{\sigma\sqrt{k}}$$

and D'_n denotes the complement of the set D_n. By the Chebyshev's inequality and the assumptions on EN_n and $V(N_n)$, it follows that $P(D'_n) \to 0$. By Theorem 1.1, it follows that

$$|P(S_n - n\mu \le x\sigma\sqrt{n}) - \Phi(x)| \to 0$$

as $n \to \infty$ for any $x \in R$ and the convergence is uniform in $x \in R$ as the limiting distribution function $\Phi(.)$ is continuous. Hence, given $\epsilon > 0$, for n and hence k large,

$$|P(S_k - k\mu \leq \sigma\sqrt{k}\, x(n,k)) - P(Z_1 \leq x(n,k))| \leq \epsilon.$$

Hence

$$d_K(T_n, T_n(Z_1)) \to 0$$

as $n \to \infty$. Observe that, as $n \to \infty$,

$$\frac{N_n}{V(S_{N_n})} \xrightarrow{P} \frac{\nu}{\nu\sigma^2 + \mu^2\tau^2}. \tag{3. 2}$$

Furthermore, since

$$\frac{V(N_n)}{V(S_{N_n})} \to \frac{\tau^2}{\nu\sigma^2 + \mu^2\tau^2}$$

as $n \to \infty$, it follows that

$$\frac{(N_n - EN_n)\mu}{\sqrt{V(S_{N_n})}} \xrightarrow{D} \frac{\mu\tau}{\sqrt{\nu\sigma^2 + \mu^2\tau^2}} Z_2 \tag{3. 3}$$

as $n \to \infty$. Applying the second inequality in Lemma 2.2 with

$$U_n = \frac{\mu(N_n - EN_n)}{\sqrt{V(S_{N_n})}},\ V = Z_1, g(U_n) = \sigma\sqrt{\frac{N_n}{V(S_{N_n})}},\ c = \sigma\sqrt{\frac{\nu}{\nu\sigma^2 + \mu^2\tau^2}}$$

and

$$U = \tau\mu Z_2\sqrt{\frac{1}{\nu\sigma^2 + \mu^2\tau^2}}\ ,$$

we conclude that

$$d_K(T_n(Z_1), T_n'(Z_1)) \to 0$$

as $n \to \infty$ by using the limits in (3.2) and (3.3). Finally, observe that

$$\begin{aligned} &d_K(T_n'(Z_1), T(Z_1, Z_2)) \\ &= \int \sup_{x \in R} \left|P(T_n'(u) \leq x) - P(T(u, Z_2) \leq x)\right| d\Phi(u) \\ &= \int \sup_{x \in R} \left|P\left(\frac{N_n - EN_n}{\sqrt{V(S_{N_n})}}\mu \leq y(x,u)\right) - P\left(\frac{\mu\tau Z_2}{\sqrt{\nu\sigma^2 + \mu^2\tau^2}} \leq y(x,u)\right)\right| d\Phi(u) \end{aligned} \tag{3. 4}$$

where

$$y(x,u) = x - u\sigma\sqrt{\frac{\nu}{\nu\sigma^2 + \mu^2\tau^2}}. \tag{3. 5}$$

Now, in view of (2.3), it follows that

$$d_K(T'_n(Z_1), T(Z_1, Z_2)) \to 0$$

as $n \to \infty$. The proof of Theorem 3.1 is now complete with the application of the triangle inequality

$$d_K(T_n, T(Z_1, Z_2)) \leq d_K(T_n, T_n(Z_1)) + d_K(T_n(Z_1), T'_n(Z_1)) + d_K(T'_n(Z_1), T(Z_1, Z_2)).$$

Remarks : (i) If the distribution function $G = \Phi$, then the random variable $T(Z_1, Z_2) = Z^* \sim N(0, 1)$.

(ii) If $\{N_n\}$ is the sum of n independent random variables $Y_1, \ldots, Y_n$ with the same mean ν and variance $\sigma^2 < \infty$, then $G = \Phi$ and the limit distribution of $\frac{S_{N_n} - E(S_{N_n})}{\sqrt{V(S_{N_n})}}$ is $N(0, 1)$.

4 Order of approximation

We shall now obtain an estimate of the order of approximation in the random central limit theorem with some additional assumptions on the random variables $\{X_n\}$ and the random indices $\{N_n\}$.

Theorem 4.1: *Let the sequence $\{X_n\}$ be a strictly stationary associated sequence of square integrable random variables satisfying the conditions (A2) and (A3). Suppose that $E|X_1|^3 < \infty$. Let the sequence $\{N_n\}$ be a sequence of nonnegative integer valued random variables such that, for each n, the random variable N_n is independent of the sequence $\{X_k, k \geq 1\}$ satisfying the assumption (A4). Let $0 < \theta < \frac{1}{2}$ and $\delta_n = n^{-\theta}$ be a sequence of positive numbers. Then there exists positive constants C_1 and C_2 such that, for all n large,*

$$(4.\ 1) \qquad d_K(T_n, T(Z_1, Z_2)) \leq C_1 n^{-\min(\theta, 1-2\theta, \frac{1}{5})} + C_2 \epsilon_n.$$

Proof : We now estimate the terms $d_K(T_n, T_n(Z_1))$, $d_K(T_n(Z_1), T'_n(Z_1))$ and $d_k(T'_n(Z_1), T(Z_1, Z_2))$ separately and then use the triangle inequality to obtain the result. We obtain an upper bound on the term $d_K(T_n, T_n(Z_1))$. Then, by the Chebyshev inequality and the bound given in Theorem 1.2, for large n, we get that

(4. 2)

$$
\begin{aligned}
d_K(T_n, T_n(Z_1)) &\leq \sum_{n\nu/2 \leq k \leq 3n\nu/2} p_{n,k} \sup_{x\in R} \left| P\left(\frac{S_k - \mu k}{\sigma\sqrt{k}} \leq x(n,k)\right) - P(Z_1 \leq x(n,k))\right| \\
&\quad + P(D'_n) \\
&\leq \frac{4V(N_n)}{(EN_n)^2} + \sum_{n\nu/2 \leq k \leq 3n\nu/2} p_{n,k} C n^{-\frac{1}{5}}. \\
&\leq \frac{C_3}{n} + \frac{C_4}{n^{\frac{1}{5}}} \leq \frac{C_5}{n^{\frac{1}{5}}}.
\end{aligned}
$$

Next, we consider estimation of the term $d_K(T_n(Z_1), T'_n(Z_1))$. By the second inequality in Lemma 2.2 with U_n, $g(U_n)$ and c as in Theorem 3.1, we get that

(4. 3)

$$
\begin{aligned}
d_K(T_n(Z_1), T'_n(Z_1)) &= \sup_{x\in R} |P(T_n(Z_1) \leq x) - P(T'_n(Z_1) \leq x)| \\
&= \sup_{x\in R} |P\left(U_n + Z_1\, g(U_n) \leq x\right) - P\left(U_n + c\, g(U_n) \leq x\right)| \\
&\leq \alpha\delta_n E|Z_1| + 2\sup_{u\in R}\left| P\left(\frac{(N_n - EN_n)\mu}{\sqrt{V(S_{N_n})}} \leq u\right) - P\left(\frac{Z_2\mu\tau}{\sqrt{\nu\sigma^2 + \mu^2\tau^2}} \leq u\right)\right| \\
&\quad + P\left(\left|\sqrt{\frac{N_n}{V(S_{N_n})}} - \sqrt{\frac{\nu}{\nu\sigma^2 + \mu^2\tau^2}}\right| > \delta_n\right) \\
&\leq C_5\delta_n + C_6\epsilon_n + P\left(\left|\sqrt{\frac{N_n}{V(S_{N_n})}} - \sqrt{\frac{\nu}{\nu\sigma^2 + \mu^2\tau^2}}\right| > \delta_n\right).
\end{aligned}
$$

Let us now estimate the term

$$
P(|\sqrt{\frac{N_n}{V(S_{N_n})}} - \sqrt{\frac{\nu}{\nu\sigma^2 + \mu^2\tau^2}}| > \delta_n) \quad .
$$

Let

$$
\gamma = \frac{\nu}{\nu\sigma^2 + \mu^2\tau^2} \quad .
$$

Observe that, for n large, using the assumption (A4) and (2.10), we get that

(4. 4)

$$
\begin{aligned}
&P(|\sqrt{\frac{N_n}{V(S_{N_n})}} - \sqrt{\frac{\nu}{\nu\sigma^2 + \mu^2\tau^2}}| > \delta_n) \\
&= P(|\sqrt{\frac{N_n}{V(S_{N_n})}} - \sqrt{\gamma}| > \delta_n)
\end{aligned}
$$

$$
\begin{aligned}
&= P(|\frac{N_n}{V(S_{N_n})} - \gamma| > \delta_n(|\sqrt{\frac{N_n}{V(S_{N_n})}} + \sqrt{\gamma}|)) \\
&\leq P(|\frac{N_n}{V(S_{N_n})} - \gamma| > \delta_n\sqrt{\gamma}) \\
&\leq P(|N_n - \gamma\, V(S_{N_n})| > \delta_n\, V(S_{N_n})\sqrt{\gamma}) \\
&\leq \frac{E(N_n - \gamma\, V(S_{N_n}))^2}{\gamma\delta_n^2 V^2(S_{N_n})} \\
&= \frac{V(N_n) + [E(N_n) - \gamma\, V(S_{N_n})]^2}{\gamma\delta_n^2 V^2(S_{N_n})} \\
&= O(\frac{n\tau^2 + [n\nu - \gamma n(\nu\sigma^2 + \mu^2\tau^2)]^2}{\gamma\delta_n^2[n^2(\nu\sigma^2 + \mu^2\tau^2)^2]}) \\
&= O(\frac{1}{n\delta_n^2}) = O(n^{2\theta-1})
\end{aligned}
$$

in view of the assumption (A4) on the sequence of random variables $\{N_n, n \geq 1\}$, and the results that the sequences $\frac{EN_n}{V(S_{N_n})}$ and $\frac{V(N_n)}{V(S_{N_n})}$ converge to finite limits as $n \to \infty$. Finally, we estimate the term $d_K(T_n'(Z_1), T(Z_1, Z_2))$. From (3.4), observe that

(4. 5)

$$
\begin{aligned}
& d_K(T_n'(Z_1), T(Z_1, Z_2)) \\
&= \int \sup_{x \in R} \left| P\left(\frac{(N_n - EN_n)\mu}{\sqrt{V(S_{N_n})}} \leq y(x,u)\right) - P\left(\frac{Z_2\mu\tau}{\sqrt{\nu\sigma^2 + \mu^2\tau^2}} \leq y(x,u)\right)\right| d\Phi(u) \\
&\leq \epsilon_n.
\end{aligned}
$$

Combining the estimates derived in (4.2)-(4.5), we get the result given in (4.1) proving Theorem 4.1.

Remarks:(i) The order of approximation depends on the rate of convergence in the central limit theorem for associated random variables as well as the rate of convergence in the weak limit theorem concerning the distribution of the random index N_n and the rate of convergence in probability of the sequence $\sqrt{\frac{N_n}{n}}$ to its limit. If $0 < \theta \leq \frac{1}{5}$, then the bound in (4.1) will be $C_1 n^{-\theta} + C_2\epsilon_n$; if $\frac{1}{5} < \theta \leq \frac{2}{5}$, then the bound in (4.1) will be $C_1 n^{-1/5} + C_2\epsilon_n$ and if $\frac{2}{5} < \theta < \frac{1}{2}$, then the bound in (4.1) will be $C_1 n^{2\theta-1} + C_2\epsilon_n$.

(ii) Let the sequence $\{Y_n, n \geq 1\}$ be a sequence of i.i.d. non-negative integer-valued random variables with $EY_1 = \nu$ and $V(Y_1) = \tau^2 < \infty$ and independent of the random variables

$\{X_k, k \geq 1\}$. Let $N_n = \sum_{j=1}^{n} Y_j$. Then $G = \Phi$ and $\epsilon_n = C_1 n^{-1/2}$. Choose $\theta = 1/3$, that is $\delta_n = n^{-1/3}$. Then $Z^* \sim N(0,1)$ and the upper bound in the order of approximation in the random central limit theorem is also $Cn^{-1/5}$.

(iii) Proofs of Theorems 3.1 and 4.1 are similar to the proofs of the corresponding results for m-dependent random variables in Prakasa Rao and Sreehari (2015). However there is an important difference. While obtaining bounds on $d_K(T_n(Z_1), T(Z_1, Z_2))$ we used a bound on $d_K(T'_n(Z_1), T(Z_1, Z_2))$ in Prakasa Rao and Sreehari (2015) as it was not possible to directly obtain a bound on $d_K(T_n(Z_1), T'_n(Z_1))$. However, in the present case, we are able to obtain a bound on $d_K(T_n(Z_1), T'_n(Z_1))$ directly.

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References :

Barbour, A.D. and Xia, A., 2006. Normal approximation for random sums, Adv. Appl. Prob. 33, 727-750.

Birkel, T., 1988. On the convergence rate in the central limit theorem for associated processes, Ann. Probab. 16, 1689-1698.

Bulinski, A.V., 1995. Rates of convergence in the central limit theorem for fields of associated random variables, Theor. Probab. Appl. 40, 136-144.

Bulinski, A.V. and Shaskin, A., 2007. Limit Theorems for Associated Random Fields and Related Systems. World Scientific, Singapore.

Dewan, I. and Prakasa Rao, B.L.S., 1997. Remarks on Berry-Esseen type bound for stationary associated random variables, Gujarat Statist. Rev. 24, 19-20.

Dewan, I. and Prakasa Rao, B.L.S., 2005. Non-uniform and uniform Berry-Esseen type bound for stationary associated sequences, J. Nonparametric Stat. 17, 217-225.

Gnedenko, B.V. and Korolev, V.Yu., 1996. Random Summation: Limit Theorems and Applications. CRC Press, Boca Raton, Fl.

Işlak , U., 2013. Asymptotic normality of random sums of m-dependent random variables, arXiv: 1303, 2386v[Math. PR].

Landers, D. and Rogge, L., 1976. The exact approximation order in the central-limit-theorem for random summation, Z. Wahr. Verw. Gebiete. 36, 269-283.

Landers, D. and Rogge, L., 1980. Sharp orders of convergence in the random central limit theorem, J. Approx. Theory. 53, 86-111.

Newman, C.M., 1980. Normal fluctuations and FKG-inequality, Commun. Math. Phys. 74, 119-128.

Oliveira, P.E., 2012. Asymptotics for Associated Random variables. Springer, Berlin.

Prakasa Rao, B.L.S., 1969. Random central limit theorem for martingales, Acta. Math. Acad. Sci. Hung. 20, 217-222.

Prakasa Rao, B.L.S., 1974. On the rate of convergence in the random central limit theorem for martingales, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. 22, 1255-1260.

Prakasa Rao, B.L.S., 1975. Remarks on the rate of convergence in the random central limit theorem for mixing sequences, Zeit. Wahr. Verw. Gebiete. 31, 157-160.

Prakasa Rao, B.L.S., 2012. Associated Sequences, Demimartingales and Nonparametric Inference. Birkhauser, Basel.

Prakasa Rao, B.L.S. and Dewan, I. 2001. Associated sequences and related inference problems, Handbook of Statistics: Stochastic Processes, Theory and Methods, ed. D.N. Shanbhag and C.R. Rao. North-Holland, Amsterdam, pp. 693-728.

Prakasa Rao, B.L.S. and Sreehari, M., 1982. Rate of convergence in the invariance principle for random sums, Sankhya Series A. 44, 144-152.

Prakasa Rao, B.L.S. and Sreehari, M., 2015. On the order of approximation in the random central limit theorem for m-dependent random variables, Probab. Math. Statist. (to appear).

Shang, Y., 2012 . A central limit theorem for randomly indexed m-dependent random variables, Filomat. 26, 713-717.

Sreehari, M., 1968. An invariance principle for random sums, Sankhya Ser. A. 30, 433-442.

Sreehari, M., 1975. Rate of convergence in the random central limit theorem, Journal M. S. University of Baroda. 24, 1-8.

Sunklodas, J.K., 2014. On the normal approximation of a binomial random sum, Lithuanian Mathematical Journal. 54, 356-365.

Tomko, J., 1971. On the estimation of the remainder term in the central limit theorem for sums of random number of summands, Theory Probab. Appl. 16, 167-175.

Wood, T.E., 1983. A Berry-Esseen theorem for associated random variables, Ann. Probab., 11, 1042-1047.