CRRAO Advanced Institute of Mathematics, Statistics and Computer Science (AIMSCS)

Research Report

Author (s): **B.L.S. Prakasa Rao**

Title of the Report: **Characterizations of probability distributions through Q-independence**

Research Report No.: **RR2016-02**

Date: **January 27, 2016**

Prof. C R Rao Road, University of Hyderabad Campus,
Gachibowli, Hyderabad-500046, INDIA.
www.crraoaimscs.org

CHARACTERIZATIONS OF PROBABILITY DISTRIBUTIONS THROUGH Q-INDEPENDENCE

B.L.S. PRAKASA RAO
CR Rao Advanced Inst. of Mathematics, Statistics and Computer Science, Hyderabad 500046, India

Abstract: We derive some characterizations of probability distributions for linear forms of Q-independent random variables.

Mathematics Subject Classification: 60E10.

Keywords: Q-independence; Q-identically distributiveness; Linear forms; Characterization of probability distributions.

1 Introduction

Kotalarski (1967) (cf. Prakasa Rao (1992), Theorem 2.1.1) proved the following result.

Theorem 1.1: Let X_1, X_2 and X_3 be three independent real-valued random variables. Let $Z_1 = X_1 - X_3$ and $Z_2 = X_2 - X_3$. If the characteristic function of the random vector (Z_1, Z_2) does not vanish, then the joint distribution of the random vector (Z_1, Z_2) determines the distributions of X_1, X_2 and X_3 up to a change of location.

Miller (1970) generalized Theorem 1.1 to random vectors. Theorem 1.1 has been extended to independent random elements taking values in a Hilbert space in Kotlarski (1966) and for random elements taking values in a locally compact abelian group in Prakasa Rao (1968). It is possible to relax the condition of non-vanishing characteristic function in Theorem 1.1 under some additional condition of analyticity of the characteristic functions. For details, see Sasvári (1986)(cf. Prakasa Rao (1992), p.15).

Rao (1971) (cf. Prakasa Rao (1992), Theorem 2.1.4) proved the following result.

Theorem 1.2: Suppose X_1, X_2 and X_3 are three independent real-valued random variables. Consider two linear forms

$$Z_1 = a_1X_1 + a_2X_2 + a_3X_3$$

and

$$Z_2 = b_1X_1 + b_2X_2 + b_3X_3$$

such that $a_i : b_i \neq a_j : b_j$ for $i \neq j$. If the characteristic function of (Z_1, Z_2) does not vanish, then the distribution of (Z_1, Z_2) determines the distribution of X_1, X_2 and X_3 up to a change of location.

Kagan and Székely (2016) introduced the notion of Q-independence and Q-identically distributed random variables and proved that the classical characterization properties of normal distribution due to Cramér (1936), Darmois-Skitovich (Darmois (1953); Skitovich (1953, 1954)), Marcinkiewicz (1938) and Vershik (1964) continue to hold for Q-independent random variables. We will now obtain some characterizations of probability distrbutions based on Q-independence property to be defined in the next section.

2 Q-independence

Let $X_1, \ldots, X_n$ be random variables and let the characteristic function of X_i be $\phi_i(t)$ for $i = 1, \ldots, n$. Following Kagan and Székely (2016), the collection $X_i, 1 \leq i \leq n$, is said to be *Q-independent* if the joint characteristic function of $(X_1, \ldots, X_n)$ can be represented as

$$\phi_{X_1,\ldots,X_n}(t_1, \ldots, t_n) \equiv E[\exp(it_1X_1 + \ldots + it_nX_n)] = \Pi_{i=1}^n \phi_i(t_i)\ \exp\{q(t_1, \ldots, t_n)\}, t_1, \ldots, t_n \in R$$

where $q(t_1, \ldots, t_n)$ is a polynomial in $t_1, \ldots, t_n$. The random variables X_j and X_k are said to be *Q-identically distributed* if

$$\phi_j(t) = \phi_k(t) \exp\{q(t)\}$$

where $q(.)$ is a polynomial.

It is known that two random variables could be Q-independent but not independent. For instance if X, Y, Z are nondegenerate independent Gaussian random variables, then $X + Y$ and $X + Z$ are Q-independent but not independent.

3 Main result

We now extend the results in Theorems 1.1 and 1.2 to Q-independent random variables.

Theorem 3.1: Let X_1, X_2 and X_3 be three Q-independent real-valued random variables. Let $Z_1 = X_1 + X_2$ and $Z_2 = X_2 + X_3$. If the characteristic function of the random vector

(Z_1, Z_2) does not vanish, then the joint distribution of the random vector (Z_1, Z_2) determines the characteristic functions of X_1, X_2 and X_3 up to multiplication by the exponentials of polynomials.

Proof: Let $\phi_{Z_1,Z_2}(t_1, t_2)$ denote the characteristic function of the bivariate random vector (Z_1, Z_2) and $\phi_i(t)$ denote the characteristic function of the random variable X_i for $i = 1, 2, 3$. Let $\phi_{X_1,X_2,X_3}(t_1, t_2, t_3)$ denote the characteristic function of the trivariate random vector (X_1, X_2, X_3). It is easy to check that

$$\begin{aligned}\phi_{Z_1,Z_2}(t_1, t_2) &= \phi_{X_1,X_2,X_3}(t_1, t_1 + t_2, t_2) \\ &= \phi_{X_1}(t_1)\phi_{X_2}(t_1 + t_2)\phi_{X_3}(t_2) \exp\{q_1(t_1, t_1 + t_2, t_2)\}\end{aligned}$$

for some polynomial $q_1(t_1, t_2, t_3)$ by the Q-independence of the random variables X_1, X_2, X_3. Suppose that $Y_i, i = 1, 2, 3$ is another set of Q-independent random variables such the joint distribution of the bivariate random vector (T_1, T_2) is the same as the joint distribution of the random vcetor (Z_1, Z_2) where $T_1 = Y_1 + Y_2$ and $T_2 = Y_2 + Y_3$. Let $\psi_{Y_i}(t)$ denote the characteristic function of the random variable Y_i for $i = 1, 2, 3$. It is easy to check that

(3. 1) $$\phi_{T_1,T_2}(t_1, t_2) = \psi_{Y_1}(t_1)\psi_{Y_2}(t_1 + t_2)\psi_{Y_3}(t_2) \exp\{q_2(t_1, t_1 + t_2, t_2)\}$$

for some polynomial $q_2(t_1, t_2, t_3)$ by the Q-independence of the random variables Y_1, Y_2, Y_3. Since the joint distributions of the random vectors (Z_1, Z_2) and (T_1, T_2) are the same and non-vanishing, by hypothesis, it follows that $\phi_{X_i}(t) \neq 0, i = 1, 2, 3$ and $\psi_{Y_i}(t) \neq 0, i = 1, 2, 3$ and

$$\phi_{X_1}(t_1)\phi_{X_2}(t_1+t_2)\phi_{X_3}(t_2) \exp\{q_1(t_1, t_1+t_2, t_2)\} = \psi_{Y_1}(t_1)\psi_{Y_2}(t_1+t_2)\psi_{Y_3}(t_2) \exp\{q_2(t_1, t_1+t_2, t_2)\}$$
(3. 2)

for $t_1, t_2 \in R$. Let $\zeta_i(t) = \log[\frac{\phi_{X_i}(t)}{\psi_{Y_i}(t)}], i = 1, 2, 3; t \in R$ where $\eta_i(t) \equiv \log \phi_{X_i}(t)$ denotes the continuous branch of the logarithm of the characteristic function $\phi_{X_i}(t)$ with $\eta_i(0) = 0$. The equations derived above imply that

(3. 3) $$\zeta_1(t_1) + \zeta_2(t_1 + t_2) + \zeta_3(t_2) = q_3(t_1, t_1 + t_2, t_2), t_1, t_2 \in R$$

where $q_3(t_1, t_1 + t_2, t_2) = q_1(t_1, t_1 + t_2, t_2) - q_2(t_1, t_1 + t_2, t_2)$ is a polynomial in t_1 and t_2. Hence

(3. 4) $$\zeta_2(t_1 + t_2) = -\zeta_1(t_1) - \zeta_3(t_2) + q_3(t_1, t_1 + t_2, t_2), t_1, t_2 \in R.$$

Applying Lemma 1.5.1 in Kagan et al. (1973), it follows that $\zeta_2(t)$ is a polynomial in t. It is easy to chek that $\zeta_i(t)$ is a polynomial in t for $i = 1, 3$ from the relation (3.3). Hence

$$\phi_{X_j}(t) = \psi_{Y_j}(t) \exp\{q_j(t)\}, t \in R, j = 1, 2, 3 \tag{3. 5}$$

where $q_j(t)$ is a polynomial in t. In particular, it follows that X_j and Y_j are Q-identically distributed for $j = 1, 2, 3$.

We now prove a result generalizing Theorem 1.2 for Q-independent random variables.

Theorem 3.3: Suppose X_1, X_2 and X_3 are three Q-independent real-valued random variables. Consider two linear forms

$$Z_1 = a_1X_1 + a_2X_2 + a_3X_3$$

and

$$Z_2 = b_1X_1 + b_2X_2 + b_3X_3$$

such that $a_i : b_i \neq a_j : b_j$ for $1 \leq i \neq j \leq 3$. If the characteristic function of (Z_1, Z_2) does not vanish, then the distribution of (Z_1, Z_2) determines the characteristic functions of X_1, X_2 and X_3 up to multiplication by the exponentials of polynomials.

Proof: Let $\phi_i(t)$ be the characteristic function of the random variable $X_i, i = 1, 2, 3$. Since the characteristic function of the bivariate random vector (Z_1, Z_2) does not vanish by hypothesis, it follows that $\phi_i(t) \neq 0$ for all $t \in R$ and for $i = 1, 2, 3$. Let $\eta_i(t) = \log \phi_i(t)$ denote the continuous branch of the logarithm of the characteristic function $\phi_i(t)$ with $\eta_i(0) = 0$. Suppose $\psi_i(t), i = 1, 2, 3$ is another set of possible characteristic functions for $X_i, i = 1, 2, 3$ respectively satisfying the hypothesis. Let $\zeta_i(t) = \log \psi_i(t)$ and

$$\gamma_i(t) = \eta_i(t) - \zeta_i(t), t \in R, i = 1, 2, 3.$$

Since the characteristic functions of the bivariate random vector (Z_1, Z_2) are the same for the choice of $\phi_i, i = 1, 2, 3$ as well as $\psi_i, i = 1, 2, 3$ and the random variables $X_i, i = 1, 2, 3$ are Q-independent, it follows that

$$\gamma_1(a_1t + b_1u)) + \gamma_2(a_2t + b_2u) + \gamma_3(a_3t + b_3u) = q(a_1t + b_1u, a_2t + b_2u, a_3t + b_3u), t, u \in R$$

where $q(t_1, t_2, t_3)$ is a polynomial in t_1, t_2, t_3. Since $a_i : b_i \neq a_j : b_j$ for $i \neq j, 1 \leq i, j \leq 3$ by hypothesis, the equation given above can be written in one of the following forms depending on the values of a_i and b_i :

(i)$\gamma_1(t+c_1u))+\gamma_2(t+c_2u)+\gamma_3(t+c_3u) = q(t+c_1u, t+c_2u, t+c_3u), t, u \in R; c_1 \neq c_2 \neq c_3 \neq 0;$

(ii)$\gamma_1(t + c_1u)) + \gamma_2(t + c_2u) = A(t) + q(t + c_1u, t + c_2u, 0); c_1 \neq c_2 \neq 0;$

(iii)$\gamma_1(t + c_1u)) = A(t) + B(u) + q(t + c_1u, 0, 0); c_1 \neq 0$

where $A(t)$ is continuous in t, $B(u)$ is continuous in u and $q(t_1, t_2, t_3)$ is a polynomial in t_1, t_2, t_3. Applying results due to Rao (1966, 1967) (cf. Prakasa Rao (1992), Lemma 2.1.2 and Lemma 2.1.3), it follows that the functions $\gamma_i(t), i = 1, 2, 3$ are polynomials of degree less than or equal to $\max(3, k)$ where k is the degree of the polynomial $q(t_1, t_2, t_3)$. This in turn implies that

$$\phi_i(t) = \psi_i(t) \exp q_i(t), i = 1, 2, 3$$

where $q_i(t)$ is a polynomial for $i = 1, 2, 3$.

The next result deals with a set of n Q-independent random variables. Proof of Theorem 3.3 is similar to that of Theorem 3.1. We omit the proof.

Theorem 3.3: Let $X_i, 1 \leq i \leq n$ be n Q-independent random variables. Define $Z_i = X_i - X_n, 1 \leq i \leq n-1$. Suppose the characteristic function of $\mathbf{Z} = (Z_1, \ldots, Z_{n-1})$ does not vanish. Then the distribution of the random vector $\mathbf{Z}$ determines the characteristic functions of $X_1, X_2, \ldots, X_n$ up to multiplication by the exponentials of polynomials.

Remarks: Rao (1971) proved that, if $X_i, 1 \leq i \leq n$ are independent random variables, then one can construct p linear forms $L_1, \ldots, L_p$ of $X_1, \ldots, X_n$ with $p(p-1)/2 \leq n \leq p(p+1)/2$ such that the joint distribution of $L_1, \ldots, L_p$ determines the distribution of $X_i, 1 \leq i \leq n$ up to Q-identical distributions. This was pointed out in Kagan and Székely (2016).

Acknowledgement : This work was supported under the scheme ”Ramanujan Chair Professor” at the CR Rao Advanced Institute of Mathematics, Statistics and Computer Science, Hyderabad 500046, India.

References:

Cramér, H. (1936) Uber eine eigenschaft der normalen verteilungsfunction, *Math. Z.*, **41**, 405-414.

Darmois, G. (1953) Analyse générale des liaisons stochastiques, *Rev. Inst. Int. Statist.*, **21**, 2-8.

Kagan, A.M., Linnik, Yu.V., and Rao, C.R. (1973) *Characterization Problems in Mathematical Statistics*, Wiley, New York.

Kagan, A.M. and Székely, Gábor J. (2016) An analytic generalization of independence and identical distributiveness, *Statistics and Probability Letters*, **110**, 244-248.

Kotlarski, I.I. (1966) On some characterizations of probability distributions in Hilbert spaces, *Annali di Math. Pura. et Appl.*, **74**, 129-134.

Kotlarski, I.I. (1967) On characterizing the gamma and the normal distribution, *Pacific J. Math.*, **20**, 69-76.

Marcinkeiwicz, J. (1938) Sur une propriété de la loi de Gauss, *Math. Z.*, **44**, 612-618.

Miller, P.G. (1970) Characterizing the distribution of three independent n-dimensional random variables X_1, X_2, X_3 having analytic characteristic functions by the joint distribution of $(X_1 + X_3, X_2 + X_3)$, *Pacific J. Math.*, **34**, 487-491.

Prakasa Rao, B.L.S. (1968) On a characterization of probability distributions on locally compact abelian groups, *Z. Wahr. verw. Geb.*, **9**, 98-100.

Prakasa Rao, B.L.S. (1992) *Identifiability in Stochastic Models : Characterization of Probability Distributions*, Academic Press, San Diego.

Rao, C.R. (1966) Characterization of the distribution of random variables in linear structural relations, *Sankhya*, **28**, 251-260.

Rao, C.R. (1967) On some characterization of the normal law, *Sankhya*, **29**, 1-14.

Rao, C.R. (1971) Characterization of probability laws by linear functions, *Sankhya, Series A*, **33**, 265-270.

Sasvári, Z. (1986) Characterizing the distribution of the random variables X_1, X_2, X_3 by the distribution of $(X_1 - X_3, X_2 - X_3)$, *Prob. Theory. and Relat. Fields*, **73**, 43-49.

Skitovich, V.P. (1953) On a property of the normal distribution, *Dokl. Akad. Nauk SSSR*, **89**, 217-219 (in Russian).

Skitovich, V.P. (1954) Linear forms in independent random variables and the normal distribution.*Dokl. Akad. Nauk SSSR, Ser. Math.*, **18**, 185-200 (in Russian).

Vershik, A.M. (1964) Some characteristic properties of Gaussian stochastic processes, *Theory Probab. Appl.*, **9**, 353-356.