CRRAO Advanced Institute of Mathematics,
Statistics and Computer Science (AIMSCS)

Research Report

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Title of the Report: **On some maximal and integral inequalities for sub-fractional Brownian motion**

Research Report No.: **RR2016-05**

Date: **August 29, 2016**

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On some maximal and integral inequalities for sub-fractional Brownian motion

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Abstract : We obtain a maximal inequality for sub-fractional Brownian motion with Hurst index $H > \frac{1}{2}$ analogous to the Burkholder-Davis-Gundy inequality for fractional Brownian motion derived by Novikov and Valkeila (Statist. Probab. Lett. 44 (1999), 47-54) and an integral inequality for Wiener integrals with respect to a sub-fractional Brownian motion with Hurst index $H > \frac{1}{2}$.

Keywords and phrases: Sub-fractional Brownian motion; Maximal inequality; Integral inequality; Wiener integral.

MSC 2010: 60G22.

1 Introduction

Fractional Brownian motion $W^H = \{W^H(t), t \geq 0\}$ has been used for modelling stochastic phenomena with long-range dependence. It is a centered Gaussian process with the covariance function

$$R_H(s,t) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$$

where $0 < H < 1$ and the constant H is called the Hurst index. The case $H = 1/2$ corresponds to the Brownian motion. FBm is the only Gaussian process which is self-similar and has stationary increments. For properties of fBm, see Samorodnitsky and Taqqu (1994), Mishura (2008) and Prakasa Rao (2010). Bojdecki et al. (2004) introduced a centered Gaussian process $\zeta^H = \{\zeta^H(t), t \geq 0\}$ called *sub-fractional Brownian motion* (sub-fBm) with the covariance function

$$C_H(s,t) = s^{2H} + t^{2H} - \frac{1}{2}[(s+t)^{2H} + |s-t|^{2H}]$$

where $0 < H < 1$. The increments of this process are not stationary and are more weakly correlated on non-overlapping intervals than those of a fBm. Tudor (2009) introduced a Wiener integral with respect to a sub-fBm. Tudor (2007 a,b, 2008, 2009) discussed some properties related to sub-fBm and its corresponding stochastic calculus. By using a fundamental martingale associated to sub-fBm, a Girsanov type theorem is obtained. Diedhiou et al. (2011) investigated parametric estimation for stochastic differential equation (SDE) driven by a sub-fBm. Mendy (2013) studied parameter estimation for sub-fractional Ornstein-Uhlenbeck process defined by the stochastic differential equation

$$dX_t = \theta X_t dt + d\zeta^H(t), t \geq 0$$

where $H > \frac{1}{2}$. Kuang and Xie (2013) studied properties of maximum likelihood estimator for sub-fBm through approximation by a random walk. Shen and Li (2014) discussed estimation for the drift of sub-fBm. Kuang and Liu (2016) discussed about the L^2-consistency and strong consistency of the maximum likelihood estimators for the sub-fBm with drift based on discrete observations. Yan et al. (2011) obtained the Ito's formula for sub-fractional Brownian motion with Hurst index $H > \frac{1}{2}$.

Our interest is to obtain some maximal and integral inequalities for sub-fBm. For an overview of maximal inequalities for fBm, see Prakasa Rao (2014).

2 Preliminaries

Bojdecki et al. (2004) noted that the process

$$\frac{1}{\sqrt{2}}[W^H(t) + W^H(-t)], t \geq 0,$$

where $\{W^H(t), -\infty < t < \infty\}$ is a fBm, is a centered Gaussian process with the same covariance function as that of a sub-fBm. This proves the existence of a sub-fBm. They proved the following result concerning properties of a sub-fBm.

Theorem 2.1: Let $\zeta^H = \{\zeta^H(t), t \geq 0\}$ be a sub-fBm. Then the following properties hold.

(i) The process ζ^H is self-similar, that is, for every $a > 0$,

$$\{\zeta^H(at), t \geq 0\} \stackrel{\Delta}{=} \{a^H \zeta^H(t), t \geq 0\}$$

in the sense that the processes, on both the sides of the equality sign, have the same finite dimensional distributions.

(ii) The process ζ^H is not Markov and it is not a semi-martingale.

(iii) For all $s, t \geq 0$, the covariance function $C_H(s,t)$ of the process ζ^H is positive for all $s > 0, t > 0$. Furthermore

$$C_H(s,t) > R_H(s,t) \text{ if } H < \frac{1}{2}$$

and

$$C_H(s,t) < R_H(s,t) \text{ if } H > \frac{1}{2}.$$

(iv) Let $\beta_H = 2 - 2^{2H-1}$. For all $s \geq 0, t \geq 0$,

$$\beta_H(t-s)^{2H} \leq E[\zeta^H(t) - \zeta^H(s)]^2 \leq (t-s)^{2H}, \text{ if } H > \frac{1}{2}$$

and

$$(t-s)^{2H} \leq E[\zeta^H(t) - \zeta^H(s)]^2 \leq \beta_H(t-s)^{2H}, \text{ if } H < \frac{1}{2}$$

and the constants in the above inequalities are sharp.

(v) The process ζ^H has continuous sample paths almost surely and, for each $0 < \epsilon < H$ and $T > 0$, there exists a random variable $K_{\epsilon,T}$ such that

$$|\zeta^H(t) - \zeta^H(s)| \leq K_{\epsilon,T}|t-s|^{H-\epsilon}, 0 \leq s,t \leq T.$$

Let $f : [0,T] \to R$ be a measurable function and $\alpha > 0$, and σ and η be real. Define the Erdeyli-Kober-type fractional integral

(2. 1)
$$(I_{T,\sigma,\eta}f)(s) = \frac{\sigma s^{\alpha\eta}}{\Gamma(\alpha)} \int_s^T \frac{t^{\sigma(1-\alpha-\eta)-1} f(t)}{(t^\sigma - s^\sigma)^{1-\alpha}} dt, s \in [0,T],$$

and

(2. 2)
$$\begin{aligned} n_H(t,s) &= \frac{\sqrt{\pi}}{2^{H-\frac{1}{2}}} I_{T,2,\frac{3-2H}{4}}(u^{H-\frac{1}{2}}) I_{[0,t)}(s) \\ &= \frac{2^{1-H}\sqrt{\pi}}{\Gamma(H-\frac{1}{2})} s^{\frac{3}{2}-H} \int_0^t (x^2 - s^2)^{H-\frac{3}{2}} dx \; I_{(0,t)}(s). \end{aligned}$$

The following theorem is due to Dzhaparidze and Van Zanten (2004) and Tudor (2009).

Theorem 2.2: The following representation holds, in distribution, for the sub-fBm ζ^H:

(2. 3)
$$\zeta_t^H \triangleq c_H \int_0^t n_H(t,s) dW_s, 0 \leq t \leq T$$

where

(2. 4)
$$c_H^2 = \frac{\Gamma(2H+1)\ \sin(\pi H)}{\pi}$$

and $\{W_t, t \geq 0\}$ is the standard Brownian motion.

Tudor (2007b) obtained the prediction formula for a sub-fBm. For any $0 < H < 1$, and $0 < a < t$,

(2. 5) $$E[\zeta_t^H | \zeta_s^H, 0 \leq s \leq a] = S_a^H + \int_0^a \psi_{a,t}(u) d\zeta_u^H$$

where

(2. 6) $$\psi_{a,t}(u) = \frac{2\ \sin(\pi(H-\frac{1}{2}))}{\pi} u(a^2-u^2)^{\frac{1}{2}-H} \int_a^t \frac{(z^2-a^2)^{H-\frac{1}{2}}}{z^2-u^2} z^{H-\frac{1}{2}} dz.$$

Let

(2. 7) $$M_t^H = d_H \int_0^t s^{\frac{1}{2}-H} dW_s$$

where

(2. 8) $$d_H = \frac{2^{H-\frac{1}{2}}}{c_H \Gamma(\frac{3}{2}-H)\sqrt{\pi}}.$$

The process $M^H = \{M_t^H, t \geq 0\}$ is a Gaussian martingale and is called the *sub-fractional fundamental martingale*. The filtration generated by this martingale is the same as the filtration $\{\mathcal{F}_t, t \geq 0\}$ generated by the sub-fBm ζ^H and the quadratic variation $< M^H, M^H >_s$ of the martingale M^H over the interval $[0, s]$ is equal to $\frac{d_H^2}{2-2H} s^{2-2H} = \lambda_H s^{2-2H}$ (say). For any measurable function $f : [0, T] \to R$ with $\int_0^T f^2(s) s^{1-2H} ds < \infty$, define the probability measure Q_f by

$$\begin{aligned} \frac{dQ_f}{dP}|_{\mathcal{F}_t} &= \exp(\int_0^t f(s) dM_s^H - \frac{1}{2}\int_0^t f^2(s) d < M^H > (s)) \\ &= \exp(\int_0^t f(s) dM_s^H - \frac{d_H^2}{2}\int_0^t f^2(s) s^{1-2H} ds). \end{aligned}$$

where P is the underlying probability measure. Let

(2. 9) $$(\psi_H f)(s) = \frac{1}{\Gamma(\frac{3}{2}-H)} I_{0,2,\frac{1}{2}-H}^{H-\frac{1}{2}} f(s)$$

where, for $\alpha > 0$,

(2. 10) $$(I_{0,\sigma,\eta} f)(s) = \frac{\sigma s^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int_0^s \frac{t^{\sigma(1+\eta)-1} f(t)}{(t^\sigma - s^\sigma)^{1-\alpha}} dt, s \in [0, T].$$

Then the following Girsanov type theorem holds for the sub-fBm process (Tudor (2009)).

Theorem 2.3: The process

$$\zeta_t^H - \int_0^t (\psi_H f)(s) ds, 0 \leq t \leq T$$

is a sub-fbm with respect to the probability measure Q_f. In particular, choosing the function $f \equiv a \in R$, it follows that the process $\{\zeta_t^H - at, 0 \leq t \leq T\}$ is a sub-fBm under the probability measure Q_f with $f \equiv a \in R$.

3 Maximal inequalities

For any process X, defined on the underlying probability space $(\Omega, \mathcal{F}, P)$, let X^* denote the supremum process defined by

$$X_t^* = \sup_{0 \leq s \leq t} |X_s|$$

whenever it is defined. Since the process ζ^H is self-similar, it follows that

$$\{\zeta^H(at), 0 \leq t \leq T\} \stackrel{\Delta}{\equiv} \{a^H \zeta^H(t), 0 \leq t \leq T\}$$

for any $a > 0$ and hence

$$\zeta^{H*}(at) \stackrel{\Delta}{\equiv} a^H \zeta^{H*}(t).$$

We have the following result as a consequence of the self-similarity of the process ζ^H.

Theorem 3.1: For any $T > 0$ and $p > 0$,

$$E[(\zeta^{H*}(T))^p] = K(H,p)T^{pH}$$

where $K(H,p) = E[(\zeta^{H*}(1))^p]$.

The following theorem is due to Burkholder-Davis-Gundy (cf. Liptser and Shiryayev (1989)).

Theorem 3.2: Let $\{N_t, \beta_t, t \geq 0\}$ be a martingale with finite quadratic variation $\{< N, N >_t, t \geq 0\}$. For any $p > 0$, and for any stopping time τ, adapted to the filtration $\{\beta_t, t \geq 0\}$, there exist positive constants c_p, C_p such that

$$c_p E[(< N, N >_\tau)^{p/2}] \leq E[(N_\tau^*)^p] \leq C_p E[(< N, N >_\tau)^{p/2}]. \quad (3.\ 1)$$

As an application of this result, we obtain the following inequality using the observation that the process $\{M_t, \mathcal{F}_t, t \geq 0\}$ is a martingale with quadratic variation $< M, M >_t = \frac{d_H^2}{2-2H} t^{2-2H}$.

Theorem 3.3: For any $p > 0$ and any stopping time τ adapted to the filtration $\{\mathcal{F}_t, t \geq 0\}$, there exist positive constants c_p, C_p such that

$$c_p \lambda_H^{p/2} E[\tau^{p(1-H)}] \leq E[(M_\tau^*)^p] \leq C_p \lambda_H^{p/2} E[\tau^{p(1-H)}]. \tag{3. 2}$$

From the results in Dzhaparidze and Van Zanten (2004) and Mendy (2013), it follows that the representation

$$W_t = \int_0^t \psi_H(t,s) d\zeta_s^H \tag{3. 3}$$

holds where $\{W_t, t \geq 0\}$ is a standard Brownian motion and

(3. 4)

$$\psi_H(t,s) = \frac{s^{H-\frac{1}{2}}}{\Gamma(\frac{3}{2}-H)} [t^{H-\frac{3}{2}}(t^2-s^2)^{\frac{1}{2}-H} - (H-\frac{3}{2}) \int_s^t (x^2-s^2)^{\frac{1}{2}-H} x^{H-\frac{3}{2}} dx] I_{(0,t)}(s).$$

Combining the equations (2.7) and (3.3), we get that

$$M_t^H = \int_0^t k_H(t,s) d\zeta_s^H \tag{3. 5}$$

where

$$k_H(t,s) = d_H s^{\frac{1}{2}-H} \psi_H(t,s) \tag{3. 6}$$

and $< M, M >_t = \lambda_H t^{2-2H}$. Following the technique in Novikov and Valkeila (1999), let

$$Y_t^H = \int_0^t s^{\frac{1}{2}-H} d\zeta_s^H, t \geq 0. \tag{3. 7}$$

Then

$$\zeta_t^H = \int_0^t s^{H-\frac{1}{2}} dY_s^H, t \geq 0 \tag{3. 8}$$

and

$$M_t^H = d_H \int_0^t k_H(t,s) s^{H-\frac{1}{2}} dY_s = d_H \int_0^t \psi_H(t,s) dY_s, t \geq 0 \tag{3. 9}$$

Equation (3.8) implies that

$$(\zeta_t^H)^* \leq 2t^\alpha (Y_t^H)^*$$

whenever $H > \frac{1}{2}$. Let $\alpha = H - \frac{1}{2}$. Solving the integral equation (3.9) as a generalized Abel integral equation with respect to the process Y^H path-wise, we can represent the process $\{Y_t^H, t \geq 0\}$ as a stochastic integral of a function $\nu_H(t,s)$ with respect to the martingale $\{M_t^H, \mathcal{F}_t, t \geq 0\}$, that is

$$Y_t^H = \int_0^t \nu_H(t,s) dM_s^H, t \geq 0. \tag{3. 10}$$

Then, it follows that

$$(Y_t^H)^* \leq \sup_{0\leq s\leq t} |\nu_H(t,s)|(M_t^H)^*, t\geq 0. \tag{3. 11}$$

Hence

$$(\zeta_t^H)^* \leq 2t^\alpha \sup_{0\leq s\leq t} |\nu_H(t,s)|(M_t^H)^*, t\geq 0. \tag{3. 12}$$

Let $\gamma_t^H = \sup_{0\leq s\leq t} |\nu_H(t,s)|$.

Applying the inequalities given above, for any stopping time τ with respect to the filtration $\{\mathcal{F}_t, t\geq 0\}$, it follows that

$$(\zeta_\tau^H)^* \leq 2\tau^\alpha \gamma_\tau^H (M_\tau^H)^*. \tag{3. 13}$$

Hence, for any $p>0$,

$$E[(\zeta_\tau^H)^*]^p \leq 2^p E[(\tau^\alpha \gamma_\tau^H)^p((M_\tau^H)^*)^p] \tag{3. 14}$$

Applying Holder's inequality with $q=\frac{H}{2\alpha}=\frac{H}{2H-1}>1$ and $r=\frac{H}{1-H}$, we get that

$$E[(\tau^\alpha\gamma_\tau^H)^p((M_\tau^H)^*)^p] \leq (E[(\tau^\alpha\gamma_\tau^H)^{pq}])^{1/q}(E[((M_\tau^H)^*)^{pr}])^{1/r}. \tag{3. 15}$$

An application of Theorem 3.2 shows that there exists a positive constant C_{pr} such that

$$E[((M_\tau^H)^*)^{pr}] \leq C_{pr}\lambda_H^{pr/2}E[\tau^{pr(1-H)}] = C_{pr}\lambda_H^{pr/2}E[\tau^{pH}] \tag{3. 16}$$

and we obtain the following theorem as a consequence of the inequalities (3.14) and (3.16).

Theorem 3.3: Let $H>\frac{1}{2}$ and τ be any stopping time adapted to filtration generated by the process $\{\zeta_t^H, t\geq 0\}$. Then, for any $p>0$, there exists a positive constant $C(p,H)$ such that

$$E[(\zeta_\tau^H)^*]^p \leq C(p,H)(E[(\tau^\alpha\gamma_\tau^H)^{pq}])^{1/q}(E[\tau^{pH}])^{1/r}. \tag{3. 17}$$

where $q=\frac{H}{2H-1}$ and $r=\frac{H}{1-H}$.

A better bound can be obtained if it is possible to derive a closed form for the function $|\nu_H(t,s)|$ and, in turn, obtain its supremum γ_t^H over any interval $[0,t]$.

4 Inequalities for Wiener integrals with respect to a sub-fBm

Tudor (2009) (cf. Mendy (2013)) has investigated properties of a Wiener integral with respect to a sub-fBm on an interval. Suppose that $\frac{1}{2}<H<1$. Let ψ denote the integral operator

$$\psi f(t) = H(2H-1)\int_0^T f(s)[|s-t|^{2H-2} - |s+t|^{2H-2}]ds \tag{4. 1}$$

and define the inner product

$$(4.\ 2) < f, g >_\psi = < f, \psi g > = H(2H-1)\int_0^T\int_0^T f(s)g(t)[|s-t|^{2H-2}-|s+t|^{2H-2}]dsdt$$

where $< . >$ denotes the usual inner product of $L^2[0,T]$. Let $L^2_\psi[0,T]$ be the space of equivalence classes of measurable functions such that $< fI_{[0,T]}, fI_{[0,T]} >_\psi < \infty$. The mapping $\zeta^H_t \to I_{[0,T]}$ can be extended to an isometry between a subspace of the Gaussian space generated by the random variables $\zeta^H_t, 0 \leq t \leq T$ and the function space $L^2_\psi[0,T]$. For $f \in L^2_\psi[0,T]$, define the integral $\int_0^T f(s)d\zeta^H_s$ as the image of the function f by this isometry. Note that the covariance function $C_H(s,t)$ the sub-fBm can be represented in the form

$$E[\zeta^H_t\zeta^H_s] = H(2H-1)\int_0^t\int_0^s [|u-v|^{2H-2}-|u+v|^{2H-2}]dudv.$$

In general, for $f, g \in L^2_\psi[0,T]$, it follows that

(4. 3)

$$E[\int_0^T f(u)d\zeta^H_u \int_0^T g(v)d\zeta^H_v] = H(2H-1)\int_0^T\int_0^T f(u)g(v)[|u-v|^{2H-2}-|u+v|^{2H-2}]dudv$$

and

$$(4.\ 4) E([\int_0^T f(u)d\zeta^H_u]^2) = H(2H-1)\int_0^T\int_0^T f(u)f(v)[|u-v|^{2H-2}-|u+v|^{2H-2}]dudv$$

We will now prove an integral inequality for a sub-fBm.

Theorem 4.1: Let ζ^H be a sub-fBm with Hurst index $H > \frac{1}{2}$. Then, for every $r > 0$, there exists a constant $c(H,r)$ such that,

$$(4.\ 5) \qquad E(|\int_0^T f(u)d\zeta^H_u|^r) \leq c(H,r)||f(u)||^r_{L^{1/H}[0,T]}.$$

We will use the following result due to Hardy and Littlewood (cf. Stein (1971), Theorem 1, p.119; Mishura (2008), Theorem 1.1.1; Samko et al. (1993)) in the proof of Theorem 4.1.

Lemma 4.2: Let $0 < \alpha < 1, 1 < p < \frac{1}{\alpha}$ and let $q = \frac{p}{1-\alpha p}$. Suppose that $f \in L_p(R)$. Then there exists a positive constant $C_{p,q,\alpha}$ such that

$$(4.\ 6) \qquad [\int_R(\int_R |f(u)||x-u|^{\alpha-1}du)^q dx]^{1/q} \leq C_{p,q,\alpha}[\int_R |f(u)|^p du]^{1/p}.$$

By replacing x by $-x$ in the above inequality, it is easy to check that

$$[\int_R (\int_R |f(u)||x+u|^{\alpha-1}du)^q dx]^{1/q} \leq C_{p,q,\alpha}[\int_R |f(u)|^p du]^{1/p} \tag{4. 7}$$

under the conditions stated in Lemma 4.2.

We will now prove Theorem 4.1.

Proof of Theorem 4.1: Since, the random variable $\int_0^T f(s)d\zeta_s^H$ is a centered Gaussian random variable, for every $r>0$, there exists a positive constant c_r such that

$$E(|\int_0^T f(u)d\zeta_u^H|^r) \leq c_r[E(|\int_0^T f(u)d\zeta_u^H|^2)]^{r/2}. \tag{4. 8}$$

In view of the equation (4.4), the inequality (4.5) will hold if

$$\int_0^T \int_0^T f(u)f(v)[|u-v|^{2H-2} - |u+v|^{2H-2}]dudv \leq c_H(\int_0^T |f(u)|^{1/H}du)^{2H}. \tag{4. 9}$$

for some constant $c_H > 0$. Choose $p = 1/H$ and $\alpha = 2H-1$ in Lemma 4.2. Note that

(4. 10)

$$\begin{aligned}\int_0^T |f(u)|(\int_0^T |f(v)||u-v|^{2H-2}dv)du &\leq (\int_0^T |f(u)|^{1/H}du)^H(\int_0^T(\int_0^T |f(v)||u-v|^{2H-2}dv)^{\frac{1}{1-H}})^{1-H}du] \\ &\leq C_{(\frac{1}{H},\frac{1}{1-H},\alpha)}[\int_0^T |f(u)|^{1/H}du]^{2H}.\end{aligned}$$

Similarly

(4. 11)

$$\begin{aligned}\int_0^T |f(u)|(\int_0^T |f(v)||u+v|^{2H-2}dv)du &\leq (\int_0^T |f(u)|^{1/H}du)^H(\int_0^T(\int_0^T |f(v)||u+v|^{2H-2})^{\frac{1}{1-H}}dv)^{1-H}du] \\ &\leq C_{(\frac{1}{H},\frac{1}{1-H},\alpha)}[\int_0^T |f(u|^{1/H}du]^{2H}.\end{aligned}$$

It is clear that

(4. 12)

$$\begin{aligned}|\int_0^T \int_0^T f(u)f(v)[|u-v|^{2H-2} - |u+v|^{2H-2}]dudv| &\leq \int_0^T \int_0^T |f(u)||f(v)||u-v|^{2H-2}dudv \\ &\quad + \int_0^T \int_0^T |f(u)||f(v)||u+v|^{2H-2}dudv.\end{aligned}$$

Combining the above inequalities, it follows that there exists a positive constant c_H such that

$$(4.\ 13)\quad |\int_0^T \int_0^T f(u)f(v)[|u-v|^{2H-2} - (u+v)^{2H-2}]dudv| \leq c_H[\int_0^T |f(u)|^{1/H}du]^{2H}$$

which in turn proves the inequality (4.5).

Acknowledgement: This work was supported under the scheme "Ramanujan Chair Professor" at the CR Rao Advanced Institute of Mathematics, Statistics and Computer science, Hyderabad, India.

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