CRRAO Advanced Institute of Mathematics, Statistics and Computer Science (AIMSCS)

Research Report

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Title of the Report: **Some maximal inequalities for fractional Brownian motion with polynomial drift**

SOME MAXIMAL INEQUALITIES FOR FRACTIONAL BROWNIAN MOTION WITH POLYNOMIAL DRIFT

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Abstract: We obtain some maximal inequalities for a centered fractional Brownian motion $B^H, H \in (\frac{1}{2}, 1)$. For a fractional Brownian motion B^H with $H \in (\frac{1}{2}, 1)$ with a polynomial drift $g(.)$, we study the asymptotic behaviour of the tail distribution function $P(\sup_{0 \le t \le T}(B_t^H + g(t)) > a)$ as $T \to \infty$ for fixed a and as $a \to \infty$ for fixed T.

1 Introduction

Asymptotic behaviour of the distribution function of the supremum of centered Gaussian processes on a compact interval and the related Borell inequality are discussed in Adler (1990). Berman (1985) derived bounds for the supremum of Gaussain processes with stationary increments over finite intervals. Michna (1998, 1999) studied the asymptotic behaviour of tail probabilities for the supremum of a fractional Brownian motion (fBm) with linear drift over a finite time interval. Debicki et al. (1998) studied the asymptotic behaviour of the supremum for Gaussian processes with linear negative drift over infinite horizon. Their results generalize and extend the earlier work in Norros (1994) and Duffield and O'Connell (1995). We obtain some maximal inequalities for centered fractional Brownian motion B^H. For a fractional Brownian motion with polynomial drift $g(.)$, we study the asymptotic behaviour of the tail distribution function $P(\sup_{0 \le t \le T}(B_t^H + g(t)) > a)$ as $T \to \infty$ for fixed a and as $a \to \infty$ for fixed T.

Let $B^H \equiv \{B_t^H, -\infty < t < \infty\}$ be a fractional Brownian motion (fBm) with Hurst index $H \in (\frac{1}{2}, 1)$, that is, a centered Gaussian process with $B_0^H = 0$ and

$$Cov(B_s^H, B_t^H) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), -\infty \le s, t < \infty.$$

It is known that the fBm B^H is self-similar, that is, for any $c > 0$,

$$\{B_{ct}^H, -\infty < t < \infty\} \stackrel{\triangle}{=} \{c^H B_t^H, -\infty < t < \infty\}$$

in the sense that the processes specified on both sides have the same finite dimensional distributions and it has stationary Gaussian increments. For a short survey of properties of a fBm, see Prakasa Rao (2010). It is known that a fBm B^H with the Hurst index $H \in (\frac{1}{2}, 1)$ has the long range dependence property and it has been used for stochastic modeling of phenomenon with long range dependence (cf. Prakasa Rao (2010)).

2 Maximal inequalities for centered fBm

The following result is due to Slepian (1962).

Theorem 2.1: (Slepian's Lemma) Let the processes $X_1 = \{X_1(t), t \geq 0\}$ and $X_2 = \{X_2(t), t \geq 0\}$ be centered Gaussian processes with $E[X_1^2(t)] = E[X_2^2(t)] = 1$. Let $\rho_1(t, s)$ and $\rho_2(t, s)$ be the covariance functions of the processes X_1 and X_2 respectively. Suppose that, for some $\delta > 0$,

$$\rho_1(t, s) \geq \rho_2(t, s), 0 \leq t, s \leq \delta.$$

Then

$$P(\sup_{0 \leq t \leq T} X_1(t) \leq u) \geq P(\sup_{0 \leq t \leq T} X_2(t) \leq u), u \in R$$

for any $0 \leq T \leq \delta$.

As a consequence of the Slepian's lemma, it follows that, if a process $\{X_t, t \geq 0\}$ is a centered almost surely continuous stationary Gaussian process with $E(X_0 X_t) \geq 0, t \geq 0$, then

$$
\begin{aligned}
P(\sup_{0 \leq t \leq T+S} X_t \leq x) &\geq P(\sup_{0 \leq t \leq T} X_t \leq x) \, P(\sup_{T \leq t \leq T+S} X_t \leq x) \\
&\quad \text{(by Slepian's lemma)} \\
&= P(\sup_{0 \leq t \leq T} X_t \leq x) \, P(\sup_{0 \leq t \leq S} X_t \leq x) \\
&\quad \text{(by stationarity)}
\end{aligned}
$$

for all $T, S > 0$ (cf. Ledoux and Talagrand (1991)). For a proof of the Slepian's lemma, see Leadbetter et al. (1983).

Let $\hat{B}^H = \{\hat{B}_t^H, t \geq 0\}$ be the scaled Brownian motion defined by $\hat{B}_t^H = W(t^{2H}), t \geq 0$ where $\{W(t), t \geq 0\}$ is the standard Wiener process. It is easy to see that the scaled Brownian

motion $\hat{B}^H$ obeys the reflection principle. Asymptotic behaviour of the tail probabilities for the supremum of scaled Brownian motion are investigated in Debicki et al. (1998).

We note that the Gaussian processes B^H and $\hat{B}^H$ satisfy the conditions of the Slepian's lemma. Applying the Slepian's lemma to the processes B^H and $\hat{B}^H$, we get that

$$P(\sup_{0 \leq t \leq T} B_t^H \geq u) \leq P(\sup_{0 \leq t \leq T} \hat{B}_t^H \geq u), u \in R.$$

Applying the reflection principle which holds for the process $\hat{B}^H$, we get that

$$P(\sup_{0 \leq t \leq T} \hat{B}_t^H \geq u) = 2\, P(\hat{B}_T^H \geq u), u \in R.$$

Hence

(2. 1) $$P(\sup_{0 \leq t \leq T} B_t^H \geq u) \leq 2\, P(\hat{B}_T^H \geq u), u \in R.$$

Observe that $\hat{B}_T^H$ is a Gaussian random variable with mean zero and variance T^{2H}. Therefore

$$P(\hat{B}_T^H \geq u) = P(Z \geq uT^{-H})$$

where Z is a standard Gaussian random variable. It is known that

$$P(Z \geq u) \leq \frac{1}{2} e^{-u^2/2},$$

and

$$P(Z \geq u) \leq \frac{1}{\sqrt{2\pi}u} e^{-u^2/2}$$

for any $u > 0$ (cf. Ito and Mckean (1965), p.17; Kutoyants (1994), p.27). Applying these inequalities, we can get the following maximal inequality for a fBm with Hurst index $H \in (\frac{1}{2}, 1)$:

Theorem 2.2: Suppose the process B^H is a centered fractional Brownian motion with Hurst index $H \in (\frac{1}{2}, 1)$. Then

$$P(\sup_{0 \leq t \leq T} B_t^H \geq u) \leq 2\, \min\{\frac{1}{2} e^{-T^{-2H} u^2/2}, \frac{1}{\sqrt{2\pi}uT^{-H}} e^{-u^2 T^{-2H}/2}\}.$$

The next result gives a bound on the expectation of the exponential of the maximum for a fBm.

Theorem 2.3: Suppose B^H is a fractional Brownian motion with Hurst index $H \in (\frac{1}{2}, 1)$. Then, for any $\lambda > 0$,

$$(2.\ 2) \qquad\qquad E[\exp\{\lambda \sup_{0 \le t \le T} |B_t^H|\}] \le 1 + \lambda\sqrt{8\pi}T^H e^{\frac{\lambda^2 T^{2H}}{2}}.$$

Proof : Let F denote the distribution function of the random variable

$$M_T^H = \sup_{0 \le t \le T} |B_t^H|.$$

Construct the Gaussian process $\hat{B}^H$ as described above. Observing that the process B^H and the process $-B^H$ are both fractional Brownian motions with Hurst index H, it is easy to see that

$$(2.\ 3) \qquad\qquad P(M_T^H \ge u) \le 4\, P(\hat{B}_T^H \ge u), u \in R$$

(cf. Muneya and Shieh (2009)). Hence, for any $\lambda > 0$,

$$
\begin{aligned}
E[\exp\{&\lambda \sup_{0 \le t \le T} |B_t^H|\}] \\
&= E[\exp\{\lambda M_T^H\}] \\
&= \int_0^\infty e^{\lambda x} F(dx) \\
&= -\int_0^\infty e^{\lambda x} d(1 - F(x)) \\
&= 1 + \lambda \int_0^\infty e^{\lambda x}(1 - F(x))\ dx \\
&= 1 + \lambda \int_0^\infty e^{\lambda x} P(M_T^H > x)\ dx \\
&\le 1 + 4\lambda \int_0^\infty e^{\lambda x} P(\hat{B}_T^H \ge x)\ dx \\
&= 1 + 4\lambda \int_0^\infty e^{\lambda x} P(T^{-H}\hat{B}_T^H \ge xT^{-H})\ dx \\
&= 1 + 4\lambda \int_0^\infty e^{\lambda x} P(Z \ge xT^{-H})\ dx
\end{aligned}
$$

where Z is a standard Gaussian random variable. Applying the inequality

$$P(Z \ge x) \le \frac{1}{2}e^{-x^2/2}, x > 0,$$

(cf. Kutoyants (1994), p.27), it follows that

$$E[\exp\{\lambda \sup_{0 \le t \le T} |B_t^H|\}]$$

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$$\le \quad 1 + 4\lambda \int_0^\infty e^{\lambda x} (\frac{1}{2} \exp[-\frac{x^2 T^{-2H}}{2}]) \, dx$$

$$\le \quad 1 + \lambda \sqrt{8\pi} T^H e^{\frac{\lambda^2 T^{2H}}{2}}.$$

Following similar methods, we now estimate

$$E[\exp\{\lambda \sup_{0 \le t \le T} (B_t^H)^2\}]$$

for $\lambda > 0$.

Theorem 2.4: Suppose B^H is a fractional Brownian motion with Hurst index $H \in (\frac{1}{2}, 1)$. Then, for any $\lambda > 0, T > 0$ such that $\lambda T^{2H} < \frac{1}{2}$,

(2. 4) $$E[\exp\{\lambda \sup_{0 \le t \le T} (B_t^H)^2\}] \le 1 + 2\lambda + 8\lambda \frac{T^{2H}}{\sqrt{1 - 2\lambda T^{2H}}}.$$

Proof: Let F denote the distribution function of the random variable

$$M_T^H = \sup_{0 \le t \le T} |B_t^H|.$$

Construct the Gaussian process $\hat{B}^H$ as described above. Observing that the process B^H and the process $-B^H$ are both fractional Brownian motions with Hurst index H, it is easy to see that

(2. 5) $$P(M_T^H \ge u) \le 4 \, P(\hat{B}_T^H \ge u), u \in R$$

(cf. Muneya and Shieh (2009)). Hence, for any $\lambda > 0$ such that $\lambda T^{2H} < \frac{1}{2}$,

$$\begin{aligned}
&E[\exp\{\lambda \sup_{0 \le t \le T} (B_t^H)^2\}] \\
&= \quad E[\exp\{\lambda (M_T^H)^2\}] \\
&= \quad \int_0^\infty e^{\lambda x^2} F(dx) \\
&= \quad -\int_0^\infty e^{\lambda x^2} d(1 - F(x)) \\
&= \quad 1 + 2\lambda \int_0^\infty x e^{\lambda x^2} (1 - F(x)) \, dx \\
&= \quad 1 + 2\lambda \int_0^\infty x e^{\lambda x^2} P(M_T^H > x) \, dx \\
&\le \quad 1 + 8\lambda \int_0^\infty x e^{\lambda x^2} P(\hat{B}_T^H \ge x) \, dx
\end{aligned}$$

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$$
= \quad 1 + 8\lambda \int_0^\infty x e^{\lambda x^2} P(T^{-H} \hat{B}_T^H \geq x T^{-H}) \, dx
$$

$$
= \quad 1 + 8\lambda \int_0^\infty x e^{\lambda x^2} P(Z \geq x T^{-H}) \, dx
$$

$$
= \quad 1 + 8\lambda \left[\int_0^1 x e^{\lambda x^2} P(Z \geq x T^{-H}) \, dx + \int_1^\infty x e^{\lambda x^2} P(Z \geq x T^{-H}) \, dx \right]
$$

where Z is a standard Gaussian random variable. Applying the inequality

$$
P(Z \geq x) \leq \frac{1}{2} e^{-x^2/2}, x > 0,
$$

(cf. Kutoyants (1994), p.27) and

$$
P(Z \geq x) \leq \frac{1}{x\sqrt{2\pi}} e^{-x^2/2}, x > 1,
$$

(cf. Ito and McKean (1965), p.17), it follows that

$$
E[\exp\{\lambda \sup_{0 \leq t \leq T} |B_t^H|^2\}]
$$

$$
\leq \quad 1 + 8\lambda \int_0^1 x e^{\lambda x^2} [\frac{1}{2} e^{-x^2 T^{-2H}/2}] \, dx
$$

$$
\quad + 8\lambda \int_1^\infty x e^{\lambda x^2} [\frac{1}{x T^{-H}\sqrt{2\pi}} e^{-x^2 T^{-2H}/2}] \, dx.
$$

$$
= \quad 1 + 4\lambda \int_0^1 x e^{\lambda x^2} e^{-\frac{x^2}{2T^{2H}}} \, dx
$$

$$
\quad + 8\lambda \frac{T^H}{\sqrt{2\pi}} [\int_1^\infty e^{\lambda x^2} e^{-x^2/2T^{2H}}] \, dx
$$

$$
= \quad 1 + 4\lambda \int_0^1 x \exp[-(\frac{1}{2T^{2H}} - \lambda)x^2] dx
$$

$$
\quad + 8\lambda \frac{T^H}{\sqrt{2\pi}} \int_1^\infty \exp[-(\frac{1}{T^{2H}} - 2\lambda)\frac{x^2}{2}] dx
$$

$$
\leq \quad 1 + 2\lambda + 8\lambda \frac{T^H}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-\frac{x^2}{2\sigma^2}} \, dx
$$

$$
= \quad 1 + 2\lambda + 8\lambda \frac{T^H}{\sqrt{2\pi}} \frac{\sqrt{2\pi}T^H}{\sqrt{1 - 2\lambda T^{2H}}}
$$

$$
= \quad 1 + 2\lambda + 8\lambda \frac{T^{2H}}{\sqrt{1 - 2\lambda T^{2H}}}
$$

for $0 < \lambda < \frac{1}{2T^{2H}}$. Here $\sigma^2 = (T^{-2H} - 2\lambda)^{-1}$.

3   Asymptotics for Maximal Inequalities for fBm with Polynomial Drift

We will now obtain some maximal inequalities for a fBm with Hurst index $H \in (\frac{1}{2}, 1)$ with polynomial drift.

Let $r > 1, 0 < \epsilon < 1, a > 0, \delta > 0$ and $k \geq 1$. We will now get a lower bound on

$$P(\sup_{0 \leq t \leq r^n \delta} [B_t^H + at^k] \leq \delta a).$$

Note that

$$
\begin{aligned}
(3.\ 1) \quad & P(\sup_{0 \leq t \leq r^n \delta} [B_t^H + at^k] \leq \delta a) \\
= \quad & P(\sup_{r^{i-1} \delta \leq t \leq r^i \delta} [B_t^H + at^k] \leq \delta a, 1 \leq i \leq n; \\
& \quad \sup_{0 \leq t \leq \epsilon \delta} [B_t^H + at^k] \leq \delta a; \text{and} \\
& \quad \sup_{\epsilon \delta \leq t \leq \delta} [B_t^H + at^k] \leq \delta a) \\
\geq \quad & P(\sup_{r^{i-1} \delta \leq t \leq r^i \delta} [B_t^H + ar^{ki} \delta^k] \leq \delta a, 1 \leq i \leq n; \\
& \quad \sup_{0 \leq t \leq \epsilon \delta} [B_t^H + \epsilon^k \delta^k] \leq \delta a; \text{and} \\
& \quad \sup_{\epsilon \delta \leq t \leq \delta} [B_t^H + a \delta^k] \leq \delta a) \\
= \quad & P(\delta^H \sup_{r^{i-1} \leq t \leq r^i} B_t^H \leq \delta a(1 - r^{ki} \delta^{k-1}), 1 \leq i \leq n; \\
& \quad \sup_{0 \leq t \leq \epsilon \delta} [B_t^H + \epsilon^k \delta^k] \leq \delta a; \text{and} \\
& \quad \sup_{\epsilon \delta \leq t \leq \delta} [B_t^H + a \delta^k] \leq \delta a) \\
& \quad (\text{by self-similarity of the process } B^H) \\
= \quad & P(\delta^H \sup_{r^{i-1} \leq t \leq r^i} B_t^H \leq \delta a(1 - r^{ki} \delta^{k-1}), 1 \leq i \leq n; \\
& \quad \delta^H \sup_{0 \leq t \leq \epsilon} B_t^H \leq \delta a(1 - \epsilon^{k-1} \delta^{k-1}); \text{and} \\
& \quad \delta^H \sup_{\epsilon \leq t \leq 1} B_t^H \leq \delta a(1 - \delta^{k-1})) \\
& \quad (\text{by self-similarity of the process } B^H) \\
\geq \quad & \Pi_{i=1}^n P(\sup_{r^{i-1} \leq t \leq r^i} B_t^H \leq \delta^{1-H} a(1 - r^{ki} \delta^{k-1}))
\end{aligned}
$$

$$\times P(\sup_{0 \leq t \leq \epsilon} B_t^H \leq \delta^{1-H} a(1 - \epsilon^{k-1}\delta^{k-1}))$$
$$\times P(\sup_{\epsilon \leq t \leq 1} B_t^H \leq \delta^{1-H} a(1 - \delta^{k-1})).$$

The last inequality follows from the Slepian's lemma (cf. Li and Shao (2004), Equation (3.3)). Let

$$i_0 = \inf\{2 \leq i \leq n : \frac{b}{[r^{i-1}(r-1)]^H} \leq 1 \text{ and } \frac{r^{ki}\delta^{k-1}b}{r^{(i-1)H}} \geq 1\}$$

where $b = \delta^{1-H}a$. Observe that i_0 depends only on a, δ, r, k and H. Let us construct a centered Gaussian process $\hat{B}^H$ with independent increments such that $\hat{B}_0^H = 0, E[\hat{B}_s^H \hat{B}_t^H] = s^{2H}$ whenever $0 \leq s \leq t \leq 1$. Let $i \in [i_0, n]$. An application of Theorem 2.1 (Slepian's lemma) shows that

$$P(\sup_{r^{i-1} \leq t \leq r^i} B_t^H \leq \delta^{1-H} a(1 - r^{ki}\delta^{k-1}))$$
$$\geq P(\sup_{r^{i-1} \leq t \leq r^i} \hat{B}_t^H \leq \delta^{1-H} a(1 - r^{ki}\delta^{k-1}))$$
$$\geq P(\sup_{r^{i-1} \leq t \leq r^i} (\hat{B}_t^H - \hat{B}_{r^{i-1}}^H) \leq b \text{ and } \hat{B}_{r^{i-1}}^H \leq -r^{ki}\delta^{k-1}b)$$
$$= P(\sup_{0 \leq t \leq (r^i-1)(r-1)} \hat{B}_t^H \leq b) P(\hat{B}_{r^{i-1}}^H \leq -r^{ki}\delta^{k-1}b)$$
$$= P[|Z_1| \leq \frac{b}{(r^{i-1}(r-1))^H}]P[Z_2 \leq -\frac{r^{ki}\delta^{k-1}b}{r^{(i-1)H}}]$$
$$= J_1 J_2 \text{ (say)}$$

where Z_1 and Z_2 are standard Gaussian random variables. For any standard Gaussian random variable Z, it is known that

$$P(|Z| \leq u) \leq \sqrt{\frac{2}{\pi}}u, 0 \leq u \leq 1$$

and

$$P(|Z| \leq u) \leq 1 - \frac{e^{-u^2/2}}{3u}, u \geq 1$$

(cf. Li (2010)). Then

$$J_1 \geq \frac{b}{3(r^{i-1}(r-1))^H}$$

and

$$J_2 \geq \frac{1}{6\frac{r^{ki}\delta^{k-1}b}{r^{(i-1)H}}} e^{-[\frac{r^{ki}\delta^{k-1}b}{r^{(i-1)H}}]^2/2}.$$

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Hence

(3. 2)
$$P(\sup_{r^{i-1} \le t \le r^i} B_t^H \le \delta^{1-H} a(1 - r^{ki}\delta^{k-1}))$$

$$\ge \frac{b}{3(r^{i-1}(r-1))^H} \frac{1}{6\frac{r^{ki}\delta^{k-1}b}{r^{(i-1)H}}} e^{-[\frac{r^{ki}\delta^{k-1}b}{r^{(i-1)H}}]^2/2}$$

$$= \frac{1}{18(r-1)^H r^{ki}\delta^{k-1}} e^{-[\frac{r^{ki}\delta^{k-1}b}{r^{(i-1)H}}]^2/2}.$$

Furthermore, for any $0 < \alpha < \beta < \infty$, and for any $x \in R$,

(3. 3)

$$P(\sup_{\alpha \le t \le \beta} B_t^H \le x)$$

$$= P(\sup_{\alpha \le t \le \beta} (B_t^H - B_\alpha^H + B_\alpha^H) \le x)$$

$$\ge P(\sup_{\alpha \le t \le \beta} (B_t^H - B_\alpha^H) \le (\beta - \alpha)|x| + (\beta - \alpha)^H \text{ and } B_\alpha^H \le x - (\beta - \alpha)|x| - (\beta - \alpha)^H)$$

$$\ge P(\sup_{\alpha \le t \le \beta} (B_t^H - B_\alpha^H) \le (\beta - \alpha)|x| + (\beta - \alpha)^H) P(B_\alpha^H \le x - (\beta - \alpha)|x| - (\beta - \alpha)^H)$$
   (by the Slepian's lemma)

$$= P(\sup_{0 \le t \le \beta - \alpha} B_t^H \le (\beta - \alpha)|x| + (\beta - \alpha)^H) P(B_\alpha^H \le x - (\beta - \alpha)|x| - (\beta - \alpha)^H)$$
   (by the stationarity of increments of the process)

$$\ge P(\sup_{0 \le t \le \beta - \alpha} \hat{B}_t^H \le (\beta - \alpha)|x| + (\beta - \alpha)^H) P(B_\alpha^H \le x - (\beta - \alpha)|x| - (\beta - \alpha)^H)$$
   (by the Slepian's lemma)

$$= P(|\hat{B}_{\beta-\alpha}^H| \le (\beta - \alpha)|x| + (\beta - \alpha)^H) P(B_\alpha^H \le x - (\beta - \alpha)|x| - (\beta - \alpha)^H)$$
   (by the reflection principle of the process $\hat{B}^H$)

$$= P(|Z_1| \le (\beta - \alpha)^{1-H}|x| + 1) P(Z_2 \le \frac{x - (\beta - \alpha)|x| - (\beta - \alpha)^H}{\alpha^H})$$

$$> 0.$$

Here Z_1 and Z_2 are standard Gaussian random variables. In addition

(3. 4)
$$P(\sup_{0 \le t \le \epsilon} B_t^H \le \delta^{1-H} a(1 - \epsilon^{k-1}\delta^{k-1}))$$

$$\ge P(\sup_{0 \le t \le \epsilon} \hat{B}_t^H \le \delta^{1-H} a(1 - \epsilon^{k-1}\delta^{k-1}))$$

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$$\begin{aligned}
&= P(|\hat{B}_\epsilon^H| \le \delta^{1-H}a(1 - \epsilon^{k-1}\delta^{k-1})) \\
&= P(|Z_3| \le \frac{\delta^{1-H}a(1 - \epsilon^{k-1}\delta^{k-1})}{\epsilon^H}) \\
&> 0
\end{aligned}$$

where Z_3 is a standard Gaussian random variable. Let

(3. 5)
$$\begin{aligned}
&C(a, \delta, r, k, H) \\
&= \Pi_{i=1}^{i_0-1} P(\sup_{r^{i-1} \le t \le r^i} B_t^H \le \delta^{1-H}a(1 - r^{ki}\delta^{k-1})) \\
&\quad \times P(\sup_{0 \le t \le \epsilon} B_t^H \le \delta^{1-H}a(1 - \epsilon^{k-1}\delta^{k-1})) \\
&\quad \times P(\sup_{\epsilon \le t \le 1} B_t^H \le \delta^{1-H}a(1 - \delta^{k-1})).
\end{aligned}$$

Inequalities (3.3) and (3.4) show that $C(a, \delta, r, k, H) > 0$. Hence

(3. 6)
$$\begin{aligned}
&P(\sup_{0 \le t \le r^n\delta} [B_t^H + at^k] \le \delta a) \\
&\ge C(a, \delta, r, k, H)\Pi_{i=i_0}^n P(\sup_{r^{i-1} \le t \le r^i} B_t^H \le \delta^{1-H}a(1 - r^{ki}\delta^{k-1})) \\
&\ge C(a, \delta, r, k, H)\Pi_{i=i_0}^n \frac{1}{18(r-1)^H r^{ki}\delta^{k-1}} e^{-[\frac{r^{ki}\delta^{k-1}b}{r^{(i-1)H}}]^2/2}.
\end{aligned}$$

Therefore, for any $r > 1$,

(3. 7)
$$\begin{aligned}
&\limsup_{T \to \infty} \frac{\log P(\sup_{0 \le t \le T}[B_t^H + at^k] \le \delta a)}{T^{2k-2H}} \\
&\ge \limsup_{n \to \infty} \frac{\log P(\sup_{0 \le t \le r^n\delta}[B_t^H + at^k] \le \delta a)}{(r^n\delta)^{2k-2H}} \\
&\ge -\frac{a^2}{2}\frac{r^{2k}}{r^{2k-2H} - 1} \equiv -\frac{a^2}{2}f(r) \text{ (say)}
\end{aligned}$$

Since the inequality derived above holds for every $r > 1$, it follows that

(3. 8)
$$\limsup_{T \to \infty} \frac{\log P(\sup_{0 \le t \le T}[B_t^H + at^k] \le \delta a)}{T^{2k-2H}} \ge -\frac{a^2}{2}\min_{r \ge 1} f(r).$$

It is easy to check that the function $f(r)$ attains its minimum over the interval $(1, \infty)$ when $r = (\frac{k}{H})^{1/(2k-2H)}$ and the minimum value is

$$(\frac{k}{H})^{2k/(2k-2H)}\frac{H}{k-H}.$$

Hence

(3. 9) $\limsup\limits_{T\to\infty} \dfrac{\log\ P(\sup_{0\le t\le T}[B_t^H + at^k] \le \delta a)}{T^{2k-2H}} \ge -\dfrac{a^2}{2}\left(\dfrac{k}{H}\right)^{2k/(2k-2H)}\dfrac{H}{k-H}.$

Let us now obtain an upper bound for

$$P(\sup_{0\le t\le T}[B_t^H + at^k] \le \delta a)$$

for large T. Observe that

(3. 10) $\begin{aligned}P(\sup_{0\le t\le T}[B_t^H + at^k] \le \delta a) &\le& P([B_T^H + aT^k] \le \delta a)\\ &=& P(B_T^H \le \delta a - aT^k)\\ &=& \dfrac{1}{2}\left(1 - P(|Z| \le \dfrac{aT^k - \delta a}{T^H})\right)\\ &\le& \dfrac{1}{2}e^{-\frac{(aT^k-\delta a)^2}{2T^{2H}}}\end{aligned}$

where Z is a standard Gaussian random variable and hence

(3. 11) $\limsup\limits_{T\to\infty} \dfrac{\log\ P(\sup_{0\le t\le T}[B_t^H + at^k] \le \delta a)}{T^{2k-2H}} \le -\dfrac{a^2}{2}.$

Combining the inequalities (3.9) and (3.11), we obtain the following theorem.

Theorem 3.1: Suppose $a > 0, \delta > 0$. Then, for any $k \ge 1$,

(3. 12) $-\dfrac{a^2}{2}\left(\dfrac{k}{H}\right)^{2k/(2k-2H)}\dfrac{H}{k-H} \le \limsup\limits_{T\to\infty} \dfrac{\log\ P(\sup_{0\le t\le T}[B_t^H + at^k] \le \delta a)}{T^{2k-2H}} \le -\dfrac{a^2}{2}.$

For $H = \frac{1}{2}$, the process B^H is the Brownian motion and the result in Theorem 3.1 generalizes Theorem 2.2 in Li (2010) for the case of the Brownian motion.

Let $g(t) = a_k t^k + a_{k-1}t^{k-1} + \ldots + a_1 t$ be a polynomial of degree k with $a_k > 0$. As a corollary to Theorem 3.1, we can obtain the following result.

Theorem 3.2: Suppose $x > 0$. Then, for any $k \ge 1$,

(3. 13) $-\dfrac{a^2}{2}\left(\dfrac{k}{H}\right)^{2k/(2k-2H)}\dfrac{H}{k-H} \le \limsup\limits_{T\to\infty} \dfrac{\log\ P(\sup_{0\le t\le T}[B_t^H + g(t)] \le x)}{T^{2k-2H}} \le -\dfrac{a^2}{2}.$

Applying the result in Theorem 3.2 for the process $-B^H$ which is again a fBm with Hurst index H, the following result follows.

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Theorem 3.3: Suppose $x > 0$. Then, for any $k \geq 1$,

$$(3.\ 14) -\frac{a^2}{2}(\frac{k}{H})^{2k/(2k-2H)}\frac{H}{k-H} \leq \limsup_{T\to\infty}\frac{\log\ P(\inf_{0\leq t\leq T}[B_t^H - g(t)] \geq -x)}{T^{2k-2H}} \leq -\frac{a^2}{2}.$$

We now discuss some other results dealing with the supremum of a fBm B^H on the interval $[0, T]$.

Theorem 3.4: Let $a > 0, \sigma > 0$ and $T > 0$. Then

$$(3.\ 15) \qquad \limsup_{\delta\to\infty}\frac{\log\ P(\sup_{0\leq t\leq T}(\sigma B_t^H + at) \geq \delta a)}{\delta^2} \leq -\frac{a^2}{2\sigma^2 T^{2H}}.$$

Proof: Let $\hat{B}^H$ be the scaled standard Brownian motion as discussed Section 1. Observe that

$$
\begin{aligned}
(3.\ 16) \qquad P(\sup_{0\leq t\leq T}(\sigma B_t^H + at) \geq \delta a) &\leq P(\sup_{0\leq t\leq T}(\sigma B_t^H + aT) \geq \delta a)\\
&= P(\sup_{0\leq t\leq T} B_t^H \geq (\delta - T)\frac{a}{\sigma})\\
&\leq P(\sup_{0\leq t\leq T}\hat{B}_t^H \geq (\delta - T)\frac{a}{\sigma})\\
&= 2\ P(\hat{B}_T^H \geq (\delta - T)\frac{a}{\sigma})\\
&= 2\ P(Z \geq T^{-H}(\delta - T)\frac{a}{\sigma})\\
&\leq \exp[-\frac{a^2(\delta - T)^2}{2\sigma^2 T^{2H}}].
\end{aligned}
$$

Hence

$$(3.\ 17) \qquad \limsup_{\delta\to\infty}\frac{\log\ P(\sup_{0\leq t\leq T}(\sigma B_t^H + at) \geq \delta a)}{\delta^2} \leq -\frac{a^2}{2\sigma^2 T^{2H}}.$$

Remarks : Duffield and O'connell (1995) proved that

$$\lim_{\delta\to\infty}\delta^{-2(1-H)}\log\ P(\sup_{t\geq 0}(B_t^H - at) > \delta) = -\inf_{c>0}c^{-2(1-H)}\frac{(c+a)^2}{2}.$$

This result deals with the tail probability of the supremum of a fBm with linear drift over an infinite horizon and gives the exact rate of convergence. Debicki et al. (1998) proved that

$$\lim_{\delta\to\infty}\delta^{-2(1-H)}\log\ P(\sup_{t\geq 0}(B_t^H - at) > \delta) = -\frac{1}{2}(\frac{a}{H})^{2H}(\frac{1}{1-H})^{2-2H}.$$

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Theorem 3.5: Let $a < 0, \sigma > 0$ and $T > 0$. Then

(3. 18)
$$\liminf_{\delta \downarrow 0} \frac{1}{-\log \delta} \log \; P(\sup_{0 \leq t \leq T} (\sigma B_t^H + at) \leq -\delta a) \geq -1.$$

Proof: Let $\hat{B}^H$ be a Gaussian Markov process with independent increments as discussed Section 1. For small $\delta > 0$, it follows from Ledoux (1996) (Equation (7.3)) that

(3. 19)
$$
\begin{aligned}
P(\sup_{0 \leq t \leq T} (\sigma B_t^H + at) \leq -\delta a) \;\; &\geq \;\; P(\sup_{0 \leq t \leq T} \sigma B_t^H \leq -\delta a) \\
&= \;\; P(\sup_{0 \leq t \leq T} B_t^H \leq -\frac{\delta a}{\sigma}) \\
&\geq \;\; P(\sup_{0 \leq t \leq T} \hat{B}_t^H \leq -\frac{\delta a}{\sigma}) \\
&= \;\; P(|\hat{B}_T^H| \leq -\frac{\delta a}{\sigma}) \\
&= \;\; P(|Z| \leq -\frac{\delta a}{T^H \sigma}) \\
&= \;\; \frac{c(a, \sigma) a}{\sigma T^H} \delta
\end{aligned}
$$

where $c(a, \sigma)$ is a constant depending only on σ and a. Hence

$$\liminf_{\delta \downarrow 0} \frac{1}{-\log \delta} \log \; P(\sup_{0 \leq t \leq T} (\sigma B_t^H + at) \leq -\delta a) \geq -1.$$

4   Limit Theorems for supremum of a fBm with Polynomial Drift

We now discuss asymptotic behaviour of the tail probabilities for the supremum of a fBm with polynomial drift over a finite time interval $[0, T]$.

Theorem 4.1: For any $k \geq 2, a > 0$ and $T > 0$,

(4. 1)
$$\lim_{\delta \to \infty} \frac{\log \; P(\sup_{0 \leq t \leq T}(\sigma B_t^H + at^k) \geq \delta a)}{\delta^2} = -\frac{a^2}{2T^{2H}}.$$

Proof : For sufficiently large δ,

$$P(\sup_{0 \leq t \leq T} (B_t^H + at^k) \geq \delta a) \;\; \geq \;\; P(\sup_{0 \leq t \leq T} B_t^H \geq \delta a)$$

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$$\geq \quad P(B_T^H \geq \delta a)$$

$$= \quad P(Z \geq \frac{\delta a}{T^H})$$

$$\geq \quad \frac{1}{6\delta a T^{-H}} e^{-\frac{\delta^2 a^2}{2T^{2H}}}.$$

Hence

(4. 2)
$$\liminf_{\delta \to \infty} \frac{\log \ P(\sup_{0 \leq t \leq T}(\sigma B_t^H + at^k) \geq \delta a)}{\delta^2} \geq -\frac{a^2}{2T^{2H}}.$$

Note that, for sufficiently large δ,

$$P(\sup_{0 \leq t \leq T}(B_t^H + at^k) \geq \delta a) \leq P(\sup_{0 \leq t \leq T}(B_t^H + aT^{k-1}t) \geq \frac{\delta}{T^{k-1}}aT^{k-1}).$$

Applying Theorem 3.4 with a replaced by aT^{k-1} and δ replaced by $\frac{\delta}{T^{k-1}}$, we get that

$$\limsup_{\delta \to \infty} \frac{\log \ P(\sup_{0 \leq t \leq T}(B_t^H + at^k) \geq \delta a)}{(\frac{\delta}{T^{k-1}})^2}$$

$$\leq \quad \limsup_{\delta \to \infty} \frac{\log \ P(\sup_{0 \leq t \leq T}(B_t^H + aT^{k-1}t) \geq \frac{\delta}{T^{k-1}}aT^{k-1})}{(\frac{\delta}{T^{k-1}})^2}$$

$$\leq \quad -\frac{(aT^{k-1})^2}{2T^{2H}}.$$

or equivalently

(4. 3)
$$\limsup_{\delta \to \infty} \frac{\log \ P(\sup_{0 \leq t \leq T}(B_t^H + at^k) \geq \delta a)}{\delta^2} \leq -\frac{a^2}{2T^{2H}}.$$

Combining the relations (4.2) and (4.3), we get that

(4. 4)
$$\lim_{\delta \to \infty} \frac{\log \ P(\sup_{0 \leq t \leq T}(B_t^H + at^k) \geq \delta a)}{\delta^2} = -\frac{a^2}{2T^{2H}}.$$

Theorem 4.2: Let $g(t) = a_k t^k + a_{k-1}t^{k-1} + \ldots + a_1 t$ with $a_k > 0$. Then, for any $T > 0$,

(4. 5)
$$\lim_{x \to \infty} \frac{\log \ P(\sup_{0 \leq t \leq T}(B_t^H + g(t)) \geq x)}{x^2} = -\frac{1}{2T^{2H}}.$$

Proof: Let $M = \sup_{0 \leq t \leq T} g(t)$ and $m = \inf_{0 \leq t \leq T} g(t)$. Then, for any $x > M$,

$$P(\sup_{0 \leq t \leq T}(B_t^H + g(t)) \geq x) \quad \leq \quad P(\sup_{0 \leq t \leq T}(B_t^H + M + t^k) \geq x)$$

$$= \quad P(\sup_{0 \leq t \leq T}(B_t^H + t^k) \geq x - M)$$

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and hence, by Theorem 4.1, it follows that

$$(4.\,6) \qquad \limsup_{x \to \infty} \frac{\log\ P(\sup_{0 \le t \le T}(B_t^H + g(t)) \ge x)}{x^2}$$

$$\le \quad \limsup_{x \to \infty} \frac{\log\ P(\sup_{0 \le t \le T}(B_t^H + t^k) \ge x - M)}{x^2}$$

$$= \quad \limsup_{x \to \infty} \frac{\log\ P(\sup_{0 \le t \le T}(B_t^H + t^k) \ge x - M)}{(x - M)^2} \frac{(x - M)^2}{x^2}$$

$$= \quad -\frac{1}{2T^{2H}}.$$

Observe that

$$P(\sup_{0 \le t \le T}(B_t^H + g(t)) \ge x) \quad \ge \quad P(\sup_{0 \le t \le T}(B_t^H + m - T^k + t^k) \ge x)$$

$$= \quad P(\sup_{0 \le t \le T}(B_t^H + t^k) \ge x - m + T^k)$$

.

Hence

$$(4.\,7) \qquad \liminf_{x \to \infty} \frac{\log\ P(\sup_{0 \le t \le T}(B_t^H + g(t)) \ge x)}{x^2}$$

$$\ge \quad \liminf_{x \to \infty} \frac{\log\ P(\sup_{0 \le t \le T}(B_t^H + t^k) \ge x - m + T^k)}{x^2}$$

$$= \quad \liminf_{x \to \infty} \frac{\log\ P(\sup_{0 \le t \le T}(B_t^H + t^k) \ge x - m + T^k)}{(x - m + T^k)^2} \frac{(x - m + T^k)^2}{x^2}$$

$$= \quad \frac{-1}{2T^{2H}}.$$

The last equality follows again from Theorem 4.1. Combining the relations (4.6) and (4.7), it follows that

$$(4.\,8) \qquad \lim_{x \to \infty} \frac{\log\ P(\sup_{0 \le t \le T}(B_t^H + g(t)) \ge x)}{x^2} = \frac{-1}{2T^{2H}}.$$

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