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SPECIAL ISSUE

ON THE FREQUENCY VARIOGRAM AND ON FREQUENCY DOMAIN METHODS FOR THE ANALYSIS OF SPATIO-TEMPORAL DATA

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In this article, we assume the spatio-temporal process to be intrinsically stationary in time and stationary in space. Our objective here is to present an alternative way, based on frequency domain methods, for modelling the data. We consider the discrete Fourier transforms (DFTs) defined for the (intrinsic) time-series data observed at several locations as our data. We use the well-known property that DFTs are asymptotically uncorrelated and distributed as complex Gaussian in deriving many results. Our objective here is to emphasize the usefulness of the DFTs in the analysis of spatio-temporal data. Under the assumption of intrinsic stationarity, we consider the estimation of frequency variogram (FV) and discuss its asymptotic sampling properties. We show that FV introduced earlier is a frequency decomposition of space–time variogram. The DFTs can be computed very fast using fast Fourier transform algorithms. Assuming that the DFTs of the incremental process satisfy a Laplacian model, an analytic expression for the space–time spectral density and an expression for the FV in terms of the spectral density function for the intrinsic stationary process are derived. The estimation of the parameters of the spectral density is also considered. A statistical test for spatial independence of spatio-temporal data is proposed.

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1. INTRODUCTION AND SUMMARY

Spatio-temporal data arises in many areas such as agriculture, geology, environmental sciences and finance. Since the data that come from these areas are functions of both time and space, any statistical method developed must take into account both spatial dependence and temporal dependence, and any interaction between these two. In the case of spatial data, the second-order spatial dependence is measured by the second-order covariance function, and if the spatial process is second-order stationary, then the second-order covariance is a function of spatial lag only. In the case of spatio-temporal data, the dependence is measured by space–time covariance function; and if the process is spatially and temporally stationary, then the covariance function is a function of the spatial lag and temporal lag. These functions are usually estimated under the assumption that the random process is spatially and temporally stationary.

An alternative second-order dependence measure is the variogram defined for both spatial processes and spatio-temporal processes. This function is well defined under the weaker assumption of intrinsic stationarity, and in view

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Dedication. Professor M. B. Priestley has made many significant contributions to the non-parametric estimation of stationary and non-stationary spectral density functions. He was one of the strong believers of the use of Fourier transforms, frequency domain methods in the analysis of time series. This article is based on the Fourier transforms and their possible application to spatio-temporal data and is written bearing in mind Professor Priestley's many important contributions in this area. We dedicate this article to him.

of this, it is widely used in geo-statistics. Its use is strongly advocated by Cressie (1993), Gringarten and Deutsch (2001) and Sherman (2011) and many others.

If the process is second-order stationary, then there is a one-to-one correspondence between the variogram and the covariance function. The estimation of the spatial covariance, spatial variogram and their asymptotic sampling properties has been considered by several authors: Cressie (1993), Yu *et al.* (2007), Stein (2012), Gneiting *et al.* (2001), Huang *et al.* (2011), Gringarten and Deutsch (2001) and Ma (2005). The literature on the estimation of space–time covariance function and the space–time variogram is not very extensive in the case of spatio-temporal random processes. The inclusion of temporal dimension complicates the estimation. The estimation and the sampling properties of the spatio-temporal covariance function have been briefly considered by Li *et al.* (2007), Cressie and Huang (1999) and Stein (2005a). See also Stein (2005b).

We may point out that instead of modelling the space–time covariance function (or its spectral density function), Cressie and Huang (1999) suggest modelling the Fourier transform of the space–time covariance function taken over space and Stein (2005b) suggests modelling its Fourier transform taken over time, which he defines as the spectra-in-time approach. The model for this function is described using a temporal spectrum given in terms of one trigonometric polynomial and a spatial correlation, the dependence of which upon the temporal frequency is specified in terms of two further trigonometric polynomials. The coefficients are estimated by maximizing the approximate likelihood and also by the spectral in time approach. In this article, we show by considering an embedded model and by modelling the discrete Fourier transforms (DFTs) using this embedded Laplacian model that under isotropy condition, the covariance function evaluated between two DFTs at the same temporal frequency gives raise to the form suggested by Stein (2005b), one part being a function of the second-order temporal spectral density and the other part given in terms of the Bessel functions. Because the covariance between the two DFTs is a real-valued function, under isotropy condition, there is no phase term in the covariance, which is a spectrum in time.

In this article, our objective is to consider the DFTs of the time series evaluated at Fourier frequencies as our data. If the observed time-series data are equally spaced, one can use the fast Fourier transform (FFT) algorithm to compute the DFTs. Using the DFTs, we model the data. Subba Rao *et al.* (2014) and Subba Rao and Terdik (2015) use the recently defined 'frequency variogram' (FV) for the estimation of the parameters of spatio-temporal covariance function of the process assuming that the DFTs satisfy a complex stochastic partial differential equation.

We show that the spatio-temporal variogram and the FV defined earlier are related. The non-parametric estimation of the FV is considered. Its sampling properties are discussed. Investigation of the sampling properties of the sample FV is much easier than that of the space–time variogram estimate. We believe that many interesting problems associated with spatio-temporal random processes can be solved using the frequency domain methods. We consider here some of these problems.

We now summarize the contents of the article. In Section 2, the space–time covariance function and space–time variogram are introduced, and their estimation, under the assumption of stationarity, is discussed in Section 3. The properties of DFTs of stationary spatial processes, spectral representation of the processes are considered in Section 4. The FV and its relation to the classical spatio-temporal variogram, and the non-parametric estimation of the FV are considered in Sections 5 and 6, and these are considered under the assumption of intrinsic stationarity of the process. Assuming that the process is intrinsically stationary, and the intrinsic process satisfies a Laplacian model, an analytic expression for the spectral density of the intrinsic process is obtained in Section 7. The estimation of the parameters of the spectral density function of the intrinsic process obtained in Section 7 is considered in Section 8. The FV and its relation to the spectral density function are also considered in Section 8. A test for spatial independence, based on the properties of complex Wishart distribution, is described in Section 9, and the test is based on the test for independence by Wahba (1971).

2. SPACE–TIME COVARIANCE FUNCTION AND THE SPACE–TIME VARIOGRAM

Let $\{Y_t(\mathbf{s}), \mathbf{s} \in \mathbb{R}^d, t \in \mathbb{Z}\}$ denote the spatio-temporal random process. Two assumptions are often made, which are important for modelling and prediction. They are that the process is second-order stationary in space and time and

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also that the process is isotropic in space. The assumption of stationarity can be sometimes unrealistic. In view of this, another weaker assumption that is often made is that the process is intrinsically stationary. We note that if the process is second-order stationary, then it implies that the process is intrinsically stationary. However, the converse is not true. We say the process $\{Y_t(\mathbf{s})\}$ is spatially, temporally second-order stationary if, for all $t \in Z, \mathbf{s} \in \mathbf{R}^d$,

$$E[Y_t(\mathbf{s})] = \mu,$$
$$\mathrm{Var}[Y_t(\mathbf{s})] = c(0,0) = \sigma_y{}^2 < \infty,$$
$$\mathrm{Cov}\,[Y_t(\mathbf{s}), Y_{t+u}(\mathbf{s}+\mathbf{h})] = c(\mathbf{h},u), \mathbf{h} \in \mathbb{R}^d, u \in \mathbb{Z}.$$

We note that $c(\mathbf{h},0)$ and $c(0,u)$ correspond to the purely spatial, purely temporal covariances respectively. Without loss of any generality, we assume that $\mu = 0$.

The random process is said to be isotropic if

$$c(\mathbf{h},u) = c\left(\|\mathbf{h}\|;u\right), \mathbf{h} \in \mathbb{R}^d, u \in \mathbb{Z},$$

where $\|\mathbf{h}\|$ is the Euclidean distance. The process is said to be fully symmetric if $c(\mathbf{h},u) = c(-\mathbf{h},u) = c(\mathbf{h},-u) = c(-\mathbf{h},-u)$ (Gneiting, 2002). The process $\{Y_t(\mathbf{s})\}$ is intrinsically spatially, temporarily stationary if the incremental process, for $u \in Z, \mathbf{h} \in \mathbf{R}^d$, $Y_t(\mathbf{s}) - Y_{t+u}(\mathbf{s}+\mathbf{h})$ satisfies the following (Cressie and Wikle, 2011, p. 315):

$$E\left((Y_t(\mathbf{s}) - Y_{t+u}(\mathbf{s}+\mathbf{h})\right) = 0,$$
$$\mathrm{Var}\left[(Y_t(\mathbf{s}) - Y_{t+u}(\mathbf{s}+\mathbf{h})\right] = \gamma(\mathbf{h},u) < \infty.$$

If $\{Y_t(\mathbf{s})\}$ is isotropic, then

$$\gamma(\mathbf{h},u) = \gamma(\|\mathbf{h}\|,u),$$

where $\gamma(\mathbf{h},u)$ is also known as the structure function (Yaglom, 1987).

The spatio-temporal variogram is defined as

$$\gamma(\mathbf{h},u) = 2\tilde{\gamma}(\mathbf{h},u) = \mathrm{Var}\left[(Y_t(\mathbf{s}) - Y_{t+u}(\mathbf{s}+\mathbf{h})\right],$$

and $\tilde{\gamma}(u,\mathbf{h})$ is defined as the semi-spatio-temporal variogram. We note that one can define the variogram under the weaker assumption of intrinsic stationarity. In other words, we do not need the assumption of stationarity of the original processes. This phenomenon of differencing in space to achieve stationarity is similar to what we have in the case of random processes with stationary increments in time, for instance, the Brownian motion.

Suppose the process $\{Y_t(\mathbf{s})\}$ is spatially and temporally stationary, and then we can show

$$\begin{aligned}\gamma(\mathbf{h},u) &= 2\left[\mathrm{Var}(Y_t(\mathbf{s})) - \mathrm{Cov}\left(Y_t(\mathbf{s}), Y_{t+u}(\mathbf{s}+\mathbf{h})\right)\right]\\ &= 2\left[c(0,0) - c(\mathbf{h},u)\right] = 2\tilde{\gamma}(\mathbf{h},u),\end{aligned}$$

and we note that there is a one-to-one correspondence between $\gamma(\mathbf{h},u)$ and $c(\mathbf{h},u)$ in the case of stationary processes. One can show that the covariance function $c(\mathbf{h},u)$ is positive semi-definite and $\gamma(\mathbf{h},u)$ is conditionally negative definite.

3. ESTIMATION OF $c(\mathbf{h},u)$ AND $\gamma(\mathbf{h},u)$

Let $\{Y_t(\mathbf{s}_i);\ i = 1,2,\ldots,m;\ t = 1,2,\ldots,n\}$ be a sample from the zero mean, stationary spatio-temporal random process $Y_t(\mathbf{s})$. We define the estimates of $c(\mathbf{h},u)$ and $\gamma(\mathbf{h},u)$ as follows (see Sherman (2011) for details). Let

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$$\hat{c}(\mathbf{h}, u) = \frac{1}{|\mathrm{N}(\mathbf{h}, u)|} \sum_{\mathrm{N}(\mathbf{h},u)} [Y_{t_i}(\mathbf{s}_i) - \bar{Y}(\mathbf{s}_i)][Y_{t_j}(\mathbf{s}_j) - \bar{Y}(\mathbf{s}_j)],$$

where

$$\bar{Y}(\mathbf{s}_i) = \frac{1}{n} \sum_{t=1}^{n} Y_t(\mathbf{s}_i),$$

and

$$\hat{\gamma}(\mathbf{h}, u) = \frac{1}{|\mathrm{N}(\mathbf{h}, u)|} \sum_{\mathrm{N}(\mathbf{h},u)} \left[Y_{t_i}(\mathbf{s}_i) - [Y_{t_j}(\mathbf{s}_j)\right]^2,$$

where $\mathrm{N}(\mathbf{h}, u) = \left\{(\mathbf{s}_i, t_i), (\mathbf{s}_j, t_j); \mathbf{s_i} - \mathbf{s_j} = \mathbf{h} \quad \text{and} \quad t_i - t_j = u\right\}$. The estimator $\hat{\gamma}(\mathbf{h}, u)$ is widely known as the Matheron estimator. In this article, we are assuming that the time-series data observed at all m locations are equally spaced and also that there are no missing values. It is interesting to investigate the properties of the estimators proposed here when these assumptions do not hold.

Under certain conditions, Li *et al.* (2007) have shown that the sample spatio-temporal covariance function defined earlier is asymptotically normal.

Based on $\hat{\gamma}(\mathbf{h}, u)$, Cressie (1993) and Huang *et al.* (2011) have proposed a weighted least squares criterion for estimating the parameters of the theoretical variogram $\gamma(\mathbf{h}, u|\theta)$, and Gneiting (2002) proposed a similar criterion for estimating the parameters based on the space–time covariance function $\hat{c}(\mathbf{h}, u)$. Subba Rao *et al.* (2014) have proposed a frequency domain method for the estimation of the parameters, which is robust against departures from Gaussianity and also computationally efficient. The method of estimation proposed by Subba Rao *et al.* (2014) is similar to the Whittle likelihood approach and is based on the FV, and the proposed criterion is easy to compute and is based on DFTs. In the following section, we define the FV and derive the sampling properties of the estimator.

4. DISCRETE FOURIER TRANSFORMS AND THE SPECTRAL REPRESENTATION OF THE PROCESS $\{Y_t(\mathbf{s})\}$

We follow the notation introduced in the paper of Subba Rao and Terdik (2015). Here we briefly highlight and summarize the results we need for our present purposes, and for further details, we refer to Subba Rao and Terdik (2015) and the books and papers cited in those papers.

We assume the random process $\{Y_t(\mathbf{s})\}$ is second-order spatially and temporally stationary. Therefore, the process has the spectral representation given by

$$Y_t(\mathbf{s}) = \int_{R^d} \int_{-\pi}^{\pi} e^{i(\mathbf{s}\cdot\boldsymbol{\lambda} + t\omega)} \mathrm{d}Z_y(\boldsymbol{\lambda}, \omega),$$

where $\mathbf{s} \cdot \boldsymbol{\lambda} = \sum_{i=1}^{d} s_i \lambda_i$ and $\int_{R^d}$ represents a d-fold multiple integral, and $Z_y(\boldsymbol{\lambda}, \omega)$ is a zero-mean complex valued random process with orthogonal increments and

$$E[\mathrm{d}Z_y(\boldsymbol{\lambda}, \omega)] = 0,$$
$$E|\mathrm{d}Z_y(\boldsymbol{\lambda}, \omega)|^2 = \mathrm{d}F_y(\boldsymbol{\lambda}, \omega),$$

where $\mathrm{d}F_y(\boldsymbol{\lambda}, \omega)$ is a spectral measure. If we assume further that $\mathrm{d}F_y(\boldsymbol{\lambda}, \omega)$ is absolutely continuous with respect to the Lebesgue measure according to the arguments $\boldsymbol{\lambda}$ and ω, then $\mathrm{d}F_y(\boldsymbol{\lambda}, \omega) = f_y(\boldsymbol{\lambda}, \omega)\mathrm{d}\boldsymbol{\lambda}\mathrm{d}\omega$, where $\mathrm{d}\boldsymbol{\lambda} = \prod_{i=1}^{d} \mathrm{d}\lambda_i$. Here $f_y(\boldsymbol{\lambda}, \omega)$ is a strictly positive, real-valued function and is defined as the spatio-temporal

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spectrum of the random process $\{Y_t(\mathbf{s})\}$, and $-\infty < \lambda_1, \lambda_2, \ldots, \lambda_d < \infty, -\pi \leq \omega \leq \pi$. In view of the orthogonality of the function $Z_y(\boldsymbol{\lambda}, \omega)$, it can be shown that

$$c(\mathbf{h}, u) = \int_{\mathbf{R}^d} \int_{-\pi}^{\pi} e^{i(\mathbf{h}\cdot\boldsymbol{\lambda}+u\omega)} f_y(\boldsymbol{\lambda}, \omega) \mathrm{d}\omega \mathrm{d}\boldsymbol{\lambda}, \tag{1}$$

and by inversion we obtain

$$f_y(\boldsymbol{\lambda}, \omega) = \frac{1}{(2\pi)^{d+1}} \sum_u \int_{-\infty}^{\infty} e^{-i(\mathbf{h}\cdot\boldsymbol{\lambda}+u\omega)} c(\mathbf{h}, u) \mathrm{d}\mathbf{h}. \tag{2}$$

From (1), we have

$$c(\mathbf{0}, u) = \int_{-\pi}^{\pi} e^{iu\omega} g_0(\omega) \mathrm{d}\omega, \tag{3}$$

where $g_0(\omega) = \int_{-\infty}^{\infty} f_y(\boldsymbol{\lambda}, \omega)\mathrm{d}\boldsymbol{\lambda}$ is the second-order temporal spectral density function of the process $\{Y_t(\mathbf{s})\}$, and in view of our assumption that the process is spatially, temporally stationary $g_0(\omega)$ is same for all the locations $\mathbf{s}$. We note $c(\mathbf{h}, u) = c(-\mathbf{h}, -u)$ and $f_y(\boldsymbol{\lambda}, \omega) = f_y(-\boldsymbol{\lambda}, -\omega)$, and $f_y(\boldsymbol{\lambda}, \omega) > 0$ for all $\boldsymbol{\lambda}$ and ω.

Here $\boldsymbol{\lambda}$ is the spatial frequency associated with the spatial coordinates $\mathbf{s}_i$ and is usually called the wave number, and ω is the temporal frequency associated with time.

Let $\{Y_t(\mathbf{s_i})\}; i = 1, 2, \ldots, m; t = 1, 2, \ldots, n$ be a sample from the zero-mean, stationary spatio-temporal random process $\{Y_t(\mathbf{s})\}$. Consider the time-series data at the location $\mathbf{s}_i$ and define the DFT

$$J_{\mathbf{s}_i}^y(\omega_k) = \frac{1}{\sqrt{2\pi n}} \sum_{t=1}^{n} Y_t(\mathbf{s}_i) e^{-it\omega_k}; \quad (i = 1, 2, \ldots, \ldots, m) \tag{4}$$

where $\omega_k = \frac{2\pi k}{n}, k = 0, 1, 2, \ldots, \left[\frac{n}{2}\right]$. We note that the DFTs can be evaluated using the FFT algorithm, and the number of operations required to calculate FFT from a time series of length n is of the order $n(\ln n)$. By inversion, we obtain from (4)

$$Y_t(\mathbf{s}) = \sqrt{\frac{n}{2\pi}} \int_{-\pi}^{\pi} e^{it\omega} J_{\mathbf{s}}^y(\omega) \mathrm{d}\omega. \tag{5}$$

The preceding representation shows that the $\{Y_t(\mathbf{s})\}$ can be decomposed into sine and cosine terms and the complex valued random variable DFT, $J_s^y(\omega)$, can be considered as the amplitude corresponding to these sine and cosine basis functions.

We will briefly summarize some well-known results associated with DFTs (Appendix), which will be required later. For details regarding properties of the DFTs for stationary processes, we refer to the books of Brillinger (2001) and Giraitis *et al.* (2012). It is well known that under some structural assumptions (Giraitis *et al.*, 2012), the DFTs $\left\{J_{\mathbf{s}}^y(\omega_k)\right\}$ evaluated at discrete Fourier frequencies ω_k are asymptotically uncorrelated and are distributed as complex normal (see, for details, Brillinger (2001) and Giraitis *et al.* (2012)).

For example, for large n, and for a specific ω_k and for a specific $\mathbf{s}$, $\left\{J_{\mathbf{s}}^y(\omega_k)\right\}$ is approximately distributed as complex normal with mean zero and variance $\{g_{\mathbf{s}}(\omega_k)\}$, which is the second-order temporal spectrum of the process at the location $\mathbf{s}$. In view of the spatial stationarity assumption, $g_{\mathbf{s}}(\omega_k)$ is the same for all locations, and we denote this common temporal spectrum by $g_0(\omega_k)$.

Let $I_{\mathbf{s}}^y(\omega_k) = |J_{\mathbf{s}}^y(\omega_k)|^2$ be the periodogram, and let $I_{\mathbf{s}_i,\mathbf{s}_j}^y(\omega_k) = J_{\mathbf{s}_i}^y(\omega_k) J_{\mathbf{s}_j}^{y*}(\omega_k)$ be the cross-periodogram between the two time series $\{Y_t(\mathbf{s}_i)\}$ and $\{Y_t(\mathbf{s}_j)\}$. In the Appendix, we summarize some properties of the periodograms (also Subba Rao and Terdik (2015)). In the following section, we define the FV and consider its estimation and also discuss the asymptotic sampling properties of the estimator proposed.

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5. FREQUENCY VARIOGRAM, PROPERTIES AND ITS ESTIMATION

As stated earlier, the variogram is used as an alternative measure of second-order dependence. It can be defined under weaker conditions, and as such, it is widely used. Although the statistical properties of the sample variogram are well studied in the case of spatial processes, the estimation and the asymptotic properties of various estimators defined for spatio-temporal processes, such as $\hat{\gamma}(\mathbf{h}, u)$ defined earlier, are not well investigated, and this could be due to the inclusion of the time dimension in the processes. To circumvent such problems, Subba Rao *et al.* (2014) have considered frequency domain approach for the statistical analysis, model construction and estimation.

Frequency variogram was introduced by Subba Rao *et al.* (2014) as an alternative to spatio-temporal variogram defined earlier and was found to be very useful in the estimation of parameters of spatio-temporal spectrum. As no inversion of high dimensional matrices is required in the estimation suggested, the computation of the minimizing criterion is easy. In this article, we consider further properties of the FV and also discuss its non-parametric estimation. We use the FV as a tool for estimating the parameters of the spatio-temporal spectrum of the intrinsic processes.

Let $\left\{J_{\mathbf{s}}^{y}(\omega_k)\right\}$ be the DFT evaluated at the Fourier frequency $\omega_k = \frac{2\pi k}{n}$; $k = 0, 1, 2, \ldots, \left[\frac{n}{2}\right]$ calculated using the time-series data $\{Y_t(\mathbf{s})\}$.

The FV is defined, for a fixed spatial lag $\mathbf{h}$ and at the location $\mathbf{s}$, as follows.

Let

$$X_t^{\mathbf{h}}(\mathbf{s}) = Y_t(\mathbf{s}) - Y_t(\mathbf{s}+\mathbf{h}), t = 1, 2, \ldots, n.$$

We have

$$E[X_t^{\mathbf{h}}(\mathbf{s})] = 0, \mathrm{Var}[X_t^{\mathbf{h}}(\mathbf{s})] = \gamma(\mathbf{h},0).$$

Define the DFT of the time series $\{X_t^{\mathbf{h}}(\mathbf{s})\}$ by

$$J_{\mathbf{s},\mathbf{s}+\mathbf{h}}^{x}(\omega) = \frac{1}{\sqrt{2\pi n}} \sum_{t=1}^{n} X_t^{\mathbf{h}}(\mathbf{s}) e^{-it\omega} = J_{\mathbf{s}}^{y}(\omega) - J_{\mathbf{s}+\mathbf{h}}^{y}(\omega),$$

and the periodogram by

$$I_{\mathbf{s},\mathbf{s}+\mathbf{h}}^{x}(\omega) \quad = |J_{\mathbf{s},\mathbf{s}+\mathbf{h}}^{x}(\omega)|^2.$$

Definition 1.

$$\begin{aligned} G_{\mathbf{s},\mathbf{s}+\mathbf{h}}^{x}(\omega) &= 2\tilde{G}_{\mathbf{s},\mathbf{s}+\mathbf{h}}^{x}(\omega) \\ &= E|J_{\mathbf{s}}^{y}(\omega) - J_{\mathbf{s}+\mathbf{h}}^{y}(\omega)|^2 \\ &= E[I_{\mathbf{s},\mathbf{s}+\mathbf{h}}^{x}(\omega)], \end{aligned}$$

for all $|\omega| \leq \pi$. Subba Rao *et al.* (2014) defined $G_{\mathbf{s},\mathbf{s}+\mathbf{h}}^{x}(\omega)$ as the FV.

We note that $J_{\mathbf{s},\mathbf{s}+\mathbf{h}}^{x}(\omega)$ is the DFT of the incremental random process $\left\{X_t^{\mathbf{h}}(\mathbf{s})\right\}$. If the incremental process defined is spatially intrinsically stationary, and also temporally stationary, then the DFTs $\left\{J_{\mathbf{s},\mathbf{s}+\mathbf{h}}^{x}(\omega_k)\right\}$ are asymptotically uncorrelated and distributed as complex Gaussian (Brillinger, 2001; Giraitis *et al.*, 2012). These functions are well defined, and no assumptions of spatial, temporal stationarity of the process $\{Y_t(\mathbf{s})\}$ are required. The FV $G_{\mathbf{s},\mathbf{s}+\mathbf{h}}^{x}(\omega)$ can be used as a measure of dissimilarity between the two random process $\{Y_t(\mathbf{s})\}$ and $\{Y_t(\mathbf{s}+\mathbf{h})\}$ at the frequency ω. As one would expect, this measure to increase as the spatial lag $\|\mathbf{h}\|$ increases and tends to zero as $||\mathbf{h}|| \to \mathbf{0}$. Some further comments on FV are in order.

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Remark 1. The FV given by $G^x_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega)$ is well defined and defined under the weaker condition of intrinsic stationarity.

Remark 2. If the intrinsic process $\{X^{\mathbf{h}}_t(\mathbf{s})\}$ is spatially and temporally stationary, its second-order periodogram $I^x_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega)$ is asymptotically an unbiased estimator of the temporal spectrum of the intrinsic process $\{X^{\mathbf{h}}_t(\mathbf{s})\}$. In view of the assumption of the spatial stationarity of the intrinsic process, the second-order spectrum does not depend on the location $\mathbf{s}$. Therefore, estimating the FV is the same as estimating the second-order spectral density function of the intrinsic process $\{X^{\mathbf{h}}_t(\mathbf{s})\}$. This estimation is considered in Section 6.

In the following, we show the relationship between the spatio-temporal variogram $\gamma(\mathbf{h}, u)$ and the FV.

Proposition 1. Let

$$G^x_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega) = E|J^x_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega)|^2,$$

And then

$$\int_{-\pi}^{\pi} G^x_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega)\mathrm{d}\omega = \gamma(\mathbf{h}, 0). \tag{6}$$

Proof
An application of Parseval's theorem gives the preceding result. □

In the preceding derivation, we used the assumption that the incremental process $\{X^{\mathbf{h}}_t(\mathbf{s})\}$ is stationary temporally and spatially even though the original process $\{Y_t(\mathbf{s})\}$ may not be spatially, temporally stationary.

The preceding result (6) shows that the FV, $G^x_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega)$ is the frequency decomposition of the classical spatio-temporal variogram $\gamma(\mathbf{h}, u)$ when $u = 0$, similar to the frequency decomposition we have for the power (variance) of the stationary random process in terms of the power spectral density function. Since $\gamma(\mathbf{h}, u)$ is a measure of dissimilarity between two spatial processes separated by lag $\mathbf{h}$, $G^x_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega)$ is also a measure of dissimilarity of the two process at the frequency ω. By plotting this function as a function of ω, one can observe in which frequency band there is a large amount of lack of similarity. This information could be useful in prediction where one can predict a time series using the time-series data from other neighbourhood locations.

Proposition 2. Let $\{Y_t(\mathbf{s})\}$ be a zero-mean second-order stationary process in space and time, and let $\{J^y_{\mathbf{s}_i}(\omega)\}$ $(i = 1, 2, \ldots, m)$ be the DFTs of $\{Y_t(\mathbf{s}_i), i = 1, 2, \ldots, m\}$. Let $G^x_{\mathbf{s}_i,\mathbf{s}_j}(\omega)$ be the FV. Then

1. The covariance function $g^y_{\mathbf{s}_i,\mathbf{s}_j}(\omega) = \mathrm{cov}(J^y_{\mathbf{s}_i}(\omega), J^x_{\mathbf{s}_j}(\omega))$ is a positive semi-definite function.
2. The FV $G^x_{\mathbf{s}_i,\mathbf{s}_j}(\omega)$ is conditionally negative definite.

Proof
Consider the sum $S_1(\omega) = \sum_{i=1}^m a_i J^y_{\mathbf{s}_i}(\omega)$, where $\{a_i\}$ can be complex. Then

$$\mathrm{Var} S_1(\omega) = \sum\sum a_i a_j{}^* \mathrm{Cov}(J^y_{\mathbf{s}_i}(\omega), J^y_{\mathbf{s}_j}(\omega)) \geq 0,$$

hence result (1) of Proposition 2.
To prove the second result, assume $\sum a_i = 0$. Then we can show that

$$\begin{aligned}\sum\sum a_i a_j{}^* G^x_{\mathbf{s}_i,\mathbf{s}_j}(\omega) &= \sum\sum a_i a_j{}^* E|J^y_{\mathbf{s}_i}(\omega) - J^y_{\mathbf{s}_j}(\omega)|^2 \\ &= -2\ \mathrm{Var}[\sum_{i=1}^m a_i J^y_{\mathbf{s}_i}(\omega)] \leq 0.\end{aligned}$$

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In the preceding derivation, we used the fact that $\sum a_i = 0$ and also the second-order spectral density function does not depend on the location $\mathbf{s}_i$ because of stationarity assumption, hence result (2) of the proposition. □

5.1. Frequency Variogram and Nugget Effect

For illustration purposes, we consider the case $d = 2$. Suppose instead of observing the process $\{Y_t(\mathbf{s}), \mathbf{s} \in \mathbb{R}^2, t \in \mathbb{Z}\}$, we observe a corrupted random process $\{\tilde{Y}_t(\mathbf{s}), \mathbf{s} \in \mathbb{R}^2, t \in \mathbb{Z}\}$, where for each $\mathbf{s}$ and t,

$$\tilde{Y}_t(\mathbf{s}) = Y_t(\mathbf{s}) + \eta_t(\mathbf{s}),$$

and $\{Y_t(\mathbf{s})\}$ and $\{\eta_t(\mathbf{s})\}$ are zero-mean spatially, temporally stationary processes and $\{Y_t(\mathbf{s})\}$ and $\{\eta_t(\mathbf{s})\}$ are independent for all t and $\mathbf{s}$, it is defined as a generalized process. Further, we assume that $\{\eta_t(\mathbf{s})\}$ is a white noise process in space and time with the second-order space–time spectrum $g_\eta(\boldsymbol{\lambda}, \omega) = \frac{\sigma_\eta{}^2}{(2\pi)^3}$ for all $\boldsymbol{\lambda}$ and ω. Define the DFT of the incremental random process of $\{\tilde{Y}_t(\mathbf{s})\}$,

$$(\tilde{Y}_t(\mathbf{s}) - \tilde{Y}_t(\mathbf{s}+\mathbf{h})) = (Y_t(\mathbf{s}) - Y_t(\mathbf{s}+\mathbf{h})) + (\eta_t(\mathbf{s}) - \eta_t(\mathbf{s}+\mathbf{h}),$$

and then we have

$$\tilde{J}_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega) = J^x_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega) + J^\eta_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega), \quad |\omega| \le \pi,$$

where

$$\tilde{J}_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega) = \frac{1}{\sqrt{2\pi n}} \sum (\tilde{Y}_t(\mathbf{s}) - \tilde{Y}_t(\mathbf{s}+\mathbf{h}))e^{-i\omega t},$$
$$J^x_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega) = \frac{1}{\sqrt{2\pi n}} \sum (Y_t(\mathbf{s}) - Y_t(\mathbf{s}+\mathbf{h}))e^{-i\omega t},$$
$$J^\eta_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega) = \frac{1}{\sqrt{2\pi n}} \sum (\eta_t(\mathbf{s}) - \eta_t(\mathbf{s}+\mathbf{h}))e^{-i\omega t}.$$

Define the FV for the process $\{\tilde{Y}_t(\mathbf{s})\}$,

$$\begin{aligned} \tilde{G}_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega) &= E|\tilde{J}_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega)|^2 \\ &= E|J^x{}_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega)|^2 + E|J^\eta{}_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega)|^2 \\ &= G^x_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega) + \frac{2\sigma^2_\eta}{(2\pi)^3}. \end{aligned} \tag{7}$$

The preceding result follows because of our assumption that the random process $\{\eta_t(\mathbf{s})\}$ is a white noise. From (7), we observe that as $\|\mathbf{h}\| \to 0$, $G^x_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega) \to 0$ for all ω and, therefore, $\tilde{G}_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega) \to \frac{\sigma^2_\eta}{(2\pi)^3}$ as $\|\mathbf{h}\| \to 0$.

If we plot $\int G_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega)\mathrm{d}\omega$ as a function of $\|\mathbf{h}\|$ and if we observe a jump near the origin $\|\mathbf{h}\|$ =0, this could be due to the presence of white noise in the process. In other words, the observations are corrupted by white noise. This effect is usually called the 'nugget effect' in geo-mining literature. In the following section, we consider the estimation of $G_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega)$ when the observations are not corrupted. In practice, one uses the FFT algorithm for computing the DFTs when the time-series data are equally spaced.

We may point out that other types of nugget effects are feasible; for example, one could have a process that is temporally correlated but spatially uncorrelated. Such processes were discussed by Stein (2005b).

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6. ESTIMATION OF THE FREQUENCY VARIOGRAM UNDER THE INTRINSIC STATIONARITY

Let $\{Y_t(\mathbf{s}_i); i = 1, 2, 3, \ldots, m;\ t = 1, 2, \ldots, n\}$ be a sample from the spatio-temporal random process $\{Y_t(\mathbf{s}_i)\}$. Here we consider the estimation of FV under the assumption that the process is intrinsically stationary both spatially and temporally. We assume that the process $\{Y_t(\mathbf{s}_i)\}$ observed is not corrupted by noise.

Consider the FV $G^x_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega) = E|J^y_{\mathbf{s}}(\omega) - J^y_{\mathbf{s}+\mathbf{h}}(\omega)|^2, |\omega| \leq \pi$. We noted earlier that the FV $G^x_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega)$ is the expected value of the periodogram of the incremental process $X^{\mathbf{h}}_t(\mathbf{s}) = Y_t(\mathbf{s}) - Y_t(\mathbf{s}+\mathbf{h}), (t = 1, 2, \ldots)$. The process $\{X^{\mathbf{h}}_t(\mathbf{s})\}$ is spatially, temporally stationary when $\mathbf{h}$ is fixed. Therefore, for large n, it is well known that the periodogram is an unbiased estimator of the second-order spectral density function of the stationary process $\{X^{\mathbf{h}}_t(\mathbf{s})\}$ although it is not a consistent estimator. Therefore, our objective here is to obtain a consistent estimator of the spectrum of the incremental process $\{X^{\mathbf{h}}_t(\mathbf{s})\}$ for a given $\mathbf{h}$, using the entire sample of discrete of Fourier transforms $\{J_{\mathbf{s}_i}(\omega_k); i = 1, 2, \ldots, m\}$, for all $\omega_k = \frac{2\pi k}{n}, (k = 0, 1, \ldots, \left[\frac{n}{2}\right])$.

Let $g^x_{\mathbf{s}_i,h}(\omega)$ be the second-order spectrum of the incremental process $\{X^{\mathbf{h}}_t(\mathbf{s}_i)\}$. Since the intrinsic process is spatially stationary $g^x_{\mathbf{s}_i,\mathbf{h}}(\omega)$ does not depend on $\mathbf{s}_i$. We denote such a stationary spectrum of the intrinsic process by $g^x_{\mathbf{h}}(\omega)$.

Let Ω denote the set of all locations $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_m$, and let $\mathrm{N}(\mathbf{h})$ denote the subset of locations, such that $\mathrm{N}(\mathbf{h}) = \{\mathbf{s}_i; i = 1, 2, \ldots, m$, such that, both $\mathbf{s}_i, \mathbf{s}_i + \mathbf{h} \in \Omega\}$. $|\mathrm{N}(\mathbf{h})|$ be the number of distinct elements in the set $\mathrm{N}(\mathbf{h})$. The estimation of stationary spectrum of a time series is well known, and therefore, we discuss the estimation of $g^x_{\mathbf{h}}(\omega)$ only briefly. For details, we refer to Priestley (1981), Brillinger (2001) and Brockwell and Davis (1987).

Consider the estimator

$$\hat{g}^x_{\mathbf{h}}(\omega) = \int_{-\pi}^{\pi} W_n(\omega - \theta) \left(\frac{1}{|\mathrm{N}(\mathbf{h})|} \sum_i I^x_{\mathbf{s}_i,\mathbf{s}_i+\mathbf{h}}(\theta) \mathrm{d}\theta \right), \tag{8}$$

where the sum has taken over the set $\mathrm{N}(\mathbf{h})$; and the weight function $W_n(\theta)$, which is a real-valued even function of θ, satisfies the following assumptions. For further details, see Priestley (1981) and Brillinger (2001).

Assumptions:

1. $W_n(\theta) \geq 0$ for all n and θ.
2. $\int W_n(\theta)\mathrm{d}\theta = 1$, all n.
3. $\int W_n^2(\theta)\mathrm{d}\theta < \infty$, all n.
4. For any $\varepsilon\ (> 0)$, $W_n(\theta) \to 0$, uniformly as $n \to \infty$, for $|\theta| > \varepsilon$.

Theorem 1. Let $g^x_{\mathbf{h}}(\omega)$ be the spectral density function of the process $\{X^{\mathbf{h}}_t(\mathbf{s}_i)\}$ for all $\mathbf{s}_i$, and let $g^x_{\mathbf{s}_i,\mathbf{s}_j}(\mathbf{h}, \omega)$ be the cross-spectral density function of the process $\{X^{\mathbf{h}}_t(\mathbf{s}_i)\}$ and $\{X^{\mathbf{h}}_t(\mathbf{s}_j)\}$. Then we have

$$E(\hat{g}^x_{\mathbf{h}}(\omega)) = g^x_{\mathbf{h}}(\omega) + O(\frac{\ln n}{n}), \tag{9}$$

and

$$\lim_{n\to\infty} \mathrm{Var}(\hat{g}_h(\omega)) = \frac{1}{|\mathrm{N}(\mathbf{h})|^2} \frac{2\pi}{n} \int W_n^2(\omega - \theta) \left[\sum_{i,j} |g^x_{\mathbf{s}_i,\mathbf{s}_j}(\mathbf{h}, \theta)|^2 \right] \mathrm{d}\theta. \tag{10}$$

Proof

Take expectations of both sides of (8),

$$E\left(\hat{g}^x_h(\omega)\right) = \int W_n(\omega - \theta)(\frac{1}{|\mathrm{N}(\mathbf{h})|}) \sum_i E(I^x_{\mathbf{s}_i,\mathbf{s}_i+\mathbf{h}}(\theta))\mathrm{d}\theta,$$

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and we have

$$E(I^x_{\mathbf{s}_i,\mathbf{s}_i+\mathbf{h}}(\theta)) = g^x_{\mathbf{h}}(\theta) + O\left(\frac{\ln n}{n}\right),$$

and, therefore, we obtain

$$E\left(\hat{g}^x_{\mathbf{h}}(\omega)\right) = g^x_{\mathbf{h}}(\omega) + O\left(\frac{\ln n}{n}\right),$$

in view of Assumption 2, and the fact that $W_n(\theta)$ is approaching the Dirac delta function concentrating its mass at $\theta = 0$. Therefore, $\hat{g}^x_{\mathbf{h}}(\omega)$ is asymptotically an unbiased estimator of $g^x_{\mathbf{h}}(\omega)$. As we have noted earlier, estimating the FV is equivalent to (for large n) estimating the spectral density $g^x_{\mathbf{h}}(\omega)$ of the intrinsic process $\{X^{\mathbf{h}}_t(\mathbf{s}_i)\}$. To obtain an expression for the variance, we consider a discrete approximation of $\widehat{g^x}_{\mathbf{h}}(\omega)$. Our derivation here is heuristic, and to obtain an expression for the covariance, we assume the intrinsic process is Gaussian, even though this assumption is not essential for proving normality or consistency (Brillinger, 2001; Giraitis *et al.*, 2012). Consider the discrete approximation of (8) and take variance of both sides, and we obtain

$$\begin{aligned}\mathrm{Var}(\hat{g}^x_{\mathbf{h}}(\omega) = &\frac{1}{|\mathrm{N}(\mathbf{h})|^2}\left(\frac{2\pi}{n}\right)^2 \sum_P \sum_{P'} W_n(\omega - \theta_P) W_n(\omega - \theta_{P'}) \\ &\times \mathrm{Cov}\left(\sum_i I^x_{\mathbf{s}_i,\mathbf{s}_i+\mathbf{h}}(\theta_P), \sum_j I^x_{\mathbf{s}_j,\mathbf{s}_{j+\mathbf{h}}}(\theta_{P'})\right),\end{aligned}$$

and we have

$$\begin{aligned}&\mathrm{Cov}\left(\sum_i I^x_{\mathbf{s}_i,\mathbf{s}_i+\mathbf{h}}(\theta_P), \sum_j I^x_{\mathbf{s}_j,\mathbf{s}_j+\mathbf{h}}(\theta_{P'})\right) \\ &= \eta(\theta_p - \theta_{p'}) \sum_i \sum_j |g^x_{\mathbf{s}_i,\mathbf{s}_j}(\mathbf{h}, \theta_p)|^2 + \eta(\theta_p + \theta_{p'}) \sum_i \sum_j |g^x_{\mathbf{s}_i,\mathbf{s}_j}(\mathbf{h}, \theta_p)|^2,\end{aligned}$$

where $\eta(\theta) = \sum_{-\infty}^{\infty} \delta(\theta - 2\pi j)$ is a Dirac comb (Brillinger, 2001, Corollary 7.22). To obtain the previous expression, we used the results already well known concerning the covariance between two periodogram ordinates (Brillinger, 2001). After the substitution of this expression for the covariance and after some simplification, we obtain

$$\lim_{n\to\infty} \mathrm{Var}(\hat{g}_{\mathbf{h}}(\omega)) = \frac{1}{|\mathrm{N}(\mathbf{h})|^2}\frac{2\pi}{n}\int W_n^2(\omega - \theta)\left[\sum\sum(g^x_{\mathbf{s}_i,\mathbf{s}_j}(\mathbf{h}, \theta))^2)\right] \mathrm{d}\theta.$$

□

The preceding result shows that $\hat{g}^x_h(\omega)$ is a mean square-consistent estimator of $g^x_h(\omega)$, and as we mentioned earlier, that $g^x_h(\omega)$ is asymptotically equivalent to the FV.

Remark 3. In the derivation of the preceding results, we have only assumed that the intrinsic process is Gaussian. The assumption of Gaussianity is made only to obtain a simple expression for the variance. The result that the estimator $\hat{g}^x_{\mathbf{h}}(\omega)$ is a consistent estimator is still valid under a non-Gaussianity assumption (Brillinger, 2001).

Remark 4. It is well known that the usual Matheron estimator for the variogram $\gamma(\mathbf{h}, \mathbf{u})$ may not be stable if the data are sparse or irregularly shaped (Schabenberger and Gotway, 2005, p. 153). In such situations, it is usual to

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consider all pairs $(\mathbf{s}_i, \mathbf{s}_j)$ such that $\mathbf{s}_i - \mathbf{s}_j = \mathbf{h} \pm \triangle$, where $\triangle$ is tolerance (Cressie and Huang, 1999). The choice of $\triangle$ is arbitrary, and the derivation of the sampling properties becomes complicated.

We can show by following the preceding similar lines, that as $n \to \infty$ (Priestley, 1981),

$$\mathrm{Cov}\left(\hat{g}_{\mathbf{h}}^{x}(\omega_1), \hat{g}_{\mathbf{h}}^{x}(\omega_2)\right) = 0 \quad \text{for} \quad \omega_1 + \omega_2 \neq 0.$$

The asymptotic normality of $\hat{g}_{\mathbf{h}}^{x}(\omega)$ can be shown using the results of Hannan (1973), Taniguchi (1980) and Deo and Chen (2000).

7. COMPLEX STOCHASTIC PARTIAL DIFFERENTIAL EQUATION FOR THE INTRINSIC PROCESS AND THE SPECTRUM FOR THE FREQUENCY VARIOGRAM

In a recent paper, Subba Rao and Terdik (2015) defined a complex stochastic partial differential equation for the spatio-temporal process and obtained an analytic expression for the spectrum of the spatio-temporal process. The parametric spectrum thus obtained from the assumed model is non-separable. A spatio-temporal random process is said to be separable if its second-order space–time spectrum can be written as a product of two positive semi-definite functions, which are, in fact, space spectrum, which is a function of wave numbers $\boldsymbol{\lambda}$, and the other part corresponds to temporal spectrum corresponding to the temporal frequency ω. As we mentioned earlier, stationarity assumption may not be realistic always, and therefore, a weaker assumption that the process is intrinsically stationary is made. Here our objective is to define a model for such an intrinsic process and obtain an analytic parametric expression for the spectrum for the intrinsic process. In a later section, we consider the estimation of the parameters of the spectral function. We may note that Yaglom (1987) and Huang *et al.* (2011) and others have obtained spectra for the variogram in the case of spatial process. Yu *et al.* (2007) and Huang *et al.* (2011) have considered non-parametric estimation of the variogram.

Consider the incremental random process $X_t^{\mathbf{h}}(s) = Y_t(\mathbf{s}) - Y_t(\mathbf{s}+\mathbf{h})$, $\mathbf{s} \in \mathbb{R}^d, t \in \mathbb{Z}$. For a fixed $\mathbf{h}$, the incremental process is a function of the spatial location $\mathbf{s} \in \mathbb{R}^d$, and time $t \in \mathbb{Z}$

We consider the process $\{X_t^{\mathbf{h}}(s)\}$, which is assumed to be a zero-mean, stationary process in space and time. Define the DFT of the time series $\{X_t^{\mathbf{h}}(s)\}$,

$$J_{\mathbf{s},\mathbf{s}+\mathbf{h}}^{(x)}(\omega_k) = \frac{1}{\sqrt{2\pi n}} \sum_{t=1}^{n} X_t^{\mathbf{h}}(s) e^{it\omega_k},$$

and the DFT $J_{\mathbf{s(L)},\mathbf{s(L)}+\mathbf{h}}^{(x)}(\omega_k)$ of the time series $\{X_t^{\mathbf{h}}(\mathbf{s(L)})\}$, where, for each t, $X_t^{\mathbf{h}}(\mathbf{s(L)}) = Y_t(\mathbf{s}+\mathbf{L}) - Y_t(\mathbf{s}+\mathbf{L}+\mathbf{h})$ at the frequencies

$$\omega_k = \frac{2\pi k}{n}\ (k = 0, 1, 2, \ldots, \left[\frac{n}{2}\right]).$$

Define the covariance between two distinct Fourier transforms $J_{\mathbf{s},\mathbf{s}+\mathbf{h}}^{(x)}(\omega)$ and $J_{\mathbf{s(L)},\mathbf{s(L)}+\mathbf{h}}^{(x)}(\omega)$,

$$g_{\mathbf{s},\mathbf{s}+\mathbf{L}}^{(\mathbf{h})}(\omega) = \mathrm{Cov}(J_{\mathbf{s},\mathbf{s}+\mathbf{h}}^{(x)}(\omega),\ \ J_{\mathbf{s(L)},\mathbf{s(L)}+\mathbf{h}}^{(x)}(\omega)),$$

where $J_{\mathbf{s},\mathbf{s}+\mathbf{h}}^{(x)}(\omega),\ J_{\mathbf{s(L)},\mathbf{s(L)}+\mathbf{h}}^{(x)}(\omega)$ respectively are DFTs of the incremental processes

$$X_t^{\mathbf{h}}(s) = Y_t(\mathbf{s}) - Y_t(\mathbf{s}+\mathbf{h}), \quad \text{and} \quad X_t^{\mathbf{h}}(\mathbf{s(L)}) = Y_t(\mathbf{s}+\mathbf{L}) - Y_t(\mathbf{s}+\mathbf{L}+\mathbf{h}),$$

for $t = 1, \ldots, n, \quad \mathbf{s} = \mathbf{s}_1, \ldots, \mathbf{s}_m$ and $\mathbf{L} \in \mathbf{R}^d$. We note that in computing the preceding result, we fix $\mathbf{h}$ and consider $\{X_t^{\mathbf{h}}(\mathbf{s})\}$ as one spatio-temporal series.

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Since the process $\{X_t^{\mathbf{h}}(\mathbf{s})\}$ is a zero-mean second-order spatially, temporally stationary, it has the spectral representation.

$$X_t^{\mathbf{h}}(\mathbf{s}) = \int_{\mathbf{R}^d} \int_{-\pi}^{\pi} e^{i(\mathbf{s}\cdot\boldsymbol{\lambda}+t\omega)} \mathrm{d}\xi_X^{(\mathbf{h})}(\boldsymbol{\lambda}, \omega),$$

where $\mathrm{d}\xi_X^{(\mathbf{h})}(\boldsymbol{\lambda}, \omega)$ is a zero-mean complex random process with orthogonal increments with

$$E[\mathrm{d}\xi_X^{(\mathbf{h})}(\boldsymbol{\lambda}, \omega)] = 0,$$
$$E|\mathrm{d}\xi_X^{(\mathbf{h})}(\boldsymbol{\lambda}, \omega)|^2 = \mathrm{d}F_X{}^{(\mathbf{h})}(\boldsymbol{\lambda}, \omega) = f_X^{(\mathbf{h})}(\boldsymbol{\lambda}, \omega)\mathrm{d}\boldsymbol{\lambda}\mathrm{d}\omega.$$

We define $f_X{}^{(\mathbf{h})}(\boldsymbol{\lambda}, \omega)$ as the spectral density function of the stationary intrinsic process $\{X_t^{\mathbf{h}}(\mathbf{s})\}$. We have the following spectral representation for the DFT of the intrinsic process.

Proposition 3. Let $J_{\mathbf{s},\mathbf{s}+\mathbf{h}}^{(x)}(\omega)$ be the DFT of the stationary time series $\{X_t^{\mathbf{h}}(\mathbf{s})\}$. Then

$$J_{\mathbf{s},\mathbf{s}+\mathbf{h}}^{(x)}(\omega) = \sqrt{\frac{n}{2\pi}} \int e^{i s\cdot\boldsymbol{\lambda}} \mathrm{d}\xi_X{}^{(\mathbf{h})}(\boldsymbol{\lambda}, \omega) + o_p(1).$$

Proof
The proof is similar to the proof given in Proposition 2 of Subba Rao and Terdik (2015), and hence, the details are omitted. □

In the following, we denote the d coordinates of the location $\mathbf{s}$ by $(s_1, s_2, \ldots, s_d)$.

Theorem 2. Let $\{J_{\mathbf{s}_i,\mathbf{s}_i+\mathbf{h}}^{(x)}(\omega); i = 1, 2, \ldots, m\}$ be the DFTs of the incremental process $\{X_t^{\mathbf{h}}(\mathbf{s}_i)\}$. Let

$$\left[\sum_{i=1}^{d} \frac{\partial^2}{\partial s_i{}^2} - |P_{\mathbf{h}}(\omega, \boldsymbol{\psi})|^2\right]^{\nu} J_{\mathbf{s},\mathbf{s}+\mathbf{h}}^{(x)}(\omega) = J_{\eta_{\mathbf{s}}}^{(\mathbf{h})}(\omega), \quad |\omega| \le \pi, \tag{11}$$

where $\nu > 0$, and $J_{\eta_{\mathbf{s}}}^{(\mathbf{h})}(\omega)$ is the DFT of the space–time white noise process $\{\eta_t(\mathbf{s})\}$ and $P_{\mathbf{h}}(\omega, \boldsymbol{\psi})$ is a polynomial in ω, and it is a function of some parameter vector $\boldsymbol{\psi}$. Then the second-order space–time spectrum of the intrinsic process $\{X_t^{\mathbf{h}}(\mathbf{s})\}$ is given by

$$f_X^{(\mathbf{h})}(\boldsymbol{\lambda}, \omega) = \frac{\sigma_\eta{}^2}{(2\pi)^{d+1}} \frac{1}{\left(\sum_{i=1}^{d} \lambda_i^2 + |P_{\mathbf{h}}(\omega, \boldsymbol{\psi})|^2\right)^{2\nu}}, \tag{12}$$

and the covariance between the periodograms (which is spectrum dependent on spatial distance $\mathbf{L}$, and the temporal frequency ω) is given by

$$\begin{aligned} g_{\mathbf{s},\mathbf{s}+\mathbf{L}}^{(\mathbf{h})}(\omega) &= \mathrm{Cov}(J_{\mathbf{s},\mathbf{s}+\mathbf{h}}^{(x)}(\omega),\ J_{\mathbf{s}(\mathbf{L}),\mathbf{s}(\mathbf{L})+\mathbf{h}}^{(x)}(\omega)) \\ &= \frac{\sigma_\eta{}^2}{(2\pi)^d 2^{2\nu-1}\Gamma(2\nu)} \left(\frac{||\mathbf{L}||}{|P_{\mathbf{h}}(\omega, \boldsymbol{\psi})|}\right)^{2\nu-\frac{d}{2}} K_{2\nu-\frac{d}{2}}\left(\|\mathbf{L}\|\,|P_{\mathbf{h}}(\omega, \boldsymbol{\psi})|\right), \end{aligned} \tag{13}$$

where $\mathbf{s}(\mathbf{L}) = \mathbf{s} + \mathbf{L}$, and $K_\nu(x)$ is the modified Bessel function of the second kind of order ν. We note that in view of spatial stationarity, the right-hand-side expression does not depend on $\mathbf{s}$ and depends only on the Euclidean

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spatial distances $||\mathbf{L}||$ and $||\mathbf{h}||$. Further, as $||\mathbf{L}|| \to 0$, the temporal spectrum of the intrinsic process $\{X_t^{\mathbf{h}}(s)\}$ is given by

$$g_0^{(\mathbf{h})}(\omega) = \mathrm{Var}(J_{\mathbf{s},\mathbf{s}+\mathbf{h}}^{(x)}(\omega)) = \frac{\sigma_\eta{}^2}{(2\pi)^{\frac{d}{2}} 2^{\frac{d}{2}} \left(|P_{\mathbf{h}}(\omega, \boldsymbol{\psi})|^2\right)^{2v-\frac{d}{2}}} \frac{\Gamma(2v - \frac{d}{2})}{\Gamma(2v)}. \tag{14}$$

Proof

The proof is similar to the proof of Theorem 1 of the paper by Subba Rao and Terdik (2015) and hence is omitted.

□

We note from expression (12) for the space–time spectrum corresponding to the process satisfying model (11) that it corresponds to a non-separable process, defined earlier. We also note further that as pointed out by one reviewer, that the assumption that the random process $\{\eta_t(\mathbf{s})\}$ is a white noise process in spatial coordinate $\mathbf{s}$ is a fiction, but still, this assumption is made in the literature. More over both covariance function and the variance given earlier depend on $\mathbf{h}$ since the polynomial $P_{\mathbf{h}}(\omega, \boldsymbol{\psi})$ is related to the second-order spectral density function of the intrinsic process $X_t^{\mathbf{h}}(\mathbf{s})$. We note that $g_0^{(\mathbf{h})}(\omega)$ depends on some parameters, say, ψ. We denote this function by $g_0^{(\mathbf{h})}(\omega, \boldsymbol{\psi})$.

Proposition 4. Let $d = 2, \quad \nu = 1$ and assume $\mathbf{h}$ is fixed. Then

$$g_0^{(\mathbf{h})}(\omega, \boldsymbol{\psi}) = \frac{\sigma_\eta{}^2}{4\pi} |P_{\mathbf{h}}(\omega, \boldsymbol{\psi})|^{-2}. \tag{15}$$

The preceding result shows that the function $|P_{\mathbf{h}}(\omega, \boldsymbol{\psi})|^2$ is related to the stationary temporal spectrum of the process $\left\{X_t^{\mathbf{h}}(\mathbf{s})\right\}$. We note further that $f_X^h(\boldsymbol{\lambda}, \omega)$ is the spatio-temporal spectrum and $g_0^{(\mathbf{h})}(\omega, \boldsymbol{\psi})$ is the stationary temporal spectrum of the process $\left\{X_t^{\mathbf{h}}(\mathbf{s})\right\}$. For large n and for a fixed $\mathbf{h}$, $\mathrm{Var}(J_{\mathbf{s},\mathbf{s}+\mathbf{h}}^{(x)}(\omega)) \approx g_0^{(\mathbf{h})}(\omega, \boldsymbol{\psi}), \quad |\omega| \le \pi$. Once again, we note that the spectral density function $f_X^{(\mathbf{h})}(\boldsymbol{\lambda}, \omega)$ is non-separable.
In the preceding result, we have shown that we can obtain a parametric expression, in a closed form, for the spectral density function of the intrinsic process. The spectral density function is given by $g_0^{(\mathbf{h})}(\omega, \boldsymbol{\psi})$. In the following section, we consider the estimation of parameter vector $\underline{\boldsymbol{\psi}}$ using the DFT of the process $\{ X_t^{\mathbf{h}}(\mathbf{s})\}$.

8. ESTIMATION OF THE PARAMETERS OF THE FREQUENCY VARIOGRAM OF THE INTRINSIC PROCESS

Matheron (1963), Cressie (1993), Stein (2012), Yu *et al.* (2007) and many others have stressed the importance of the variogram in Kriging, and in view of this, several methods of estimation of the variogram in the case of spatial processes have been proposed. Yu *et al.* (2007) have proposed non-parametric estimation of the variogram, and Huang *et al.* (2011) proposed the estimation of the variogram and its spectrum. If one assumes that the intrinsic process satisfies a specific model, which have parameters that are usually unknown, then one needs to estimate the parameters of the model. In Section 7, we have obtained an expression for the spectral density function of the intrinsic process assuming that the process satisfies the model given in Theorem 2, and we have seen that the spectrum depends on the parameter vector ψ. We consider the estimation of the parameter vector ψ from the data $\left\{X_t^{\mathbf{h}}(\mathbf{s})\right\}$.

Our objective here is to estimate $\boldsymbol{\psi}$ of $g_0^{(\mathbf{h})}(\omega, \boldsymbol{\psi})$ given the DFTs $\{J_{\mathbf{s},\mathbf{s}+\mathbf{h}}^{(x)}(\omega_k); i = 1, 2, \ldots, m; k = 1, 2, \ldots, [\frac{n}{2}]$ obtained from the intrinsic processes $\{X_t^{\mathbf{h}}(\mathbf{s}_i); t = 1, 2, \ldots, n; i = 1, 2, \ldots, m\}$. Let the set $\mathrm{N}(\mathbf{h}) = \{\mathbf{s}_i; i = 1, 2, \ldots, m, \mathbf{s}_i, \mathbf{s}_i + \mathbf{h} \in \Omega\}$. If we are assuming that the DFT of the intrinsic process satisfies model (11) stated in Theorem 2, then the parameters we have to consider for the estimation are $\boldsymbol{\psi}$ of the polynomial $P_{\mathbf{h}}(\omega, \boldsymbol{\psi})$ related to the temporal spectrum $g_0{}^{(\mathbf{h})}(\omega, \boldsymbol{\psi})$ of the process$\{ X_t^{\mathbf{h}}(\mathbf{s}_i)\}$. Here we obtain the likelihood function using the DFTs, and the approach is similar to the method described in Subba Rao *et al.* (2014). We refer to the mentioned paper for details.

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Consider the DFTs $\{J^{(x)}_{\mathbf{s}_i,\mathbf{s}_i+\mathbf{h}}(\omega_k))\}$ corresponding to the time series $\{Y_t(\mathbf{s}_i)\}, \{Y_t(\mathbf{s}_i+\mathbf{h})\}$. We note that for large n, the complex valued random variable $J^{(x)}_{\mathbf{s}_i,\mathbf{s}_i+\mathbf{h}}(\omega_k)$ is asymptotically distributed as complex normal with mean zero and variance $g_0{}^{(\mathbf{h})}(\omega_k,\boldsymbol{\psi})$ (Brillinger, 2001; Giraitis *et al.*, 2012) and independent over distinct frequencies. Let $M=\left[\frac{n}{2}\right]$. Consider the M dimensional complex valued random vector

$$L_{\|\mathbf{h}\|}(\omega)=\{J^{(x)}_{\mathbf{s}_i,\mathbf{s}_i+\mathbf{h}}(\omega_1), J^{(x)}_{\mathbf{s}_i,\mathbf{s}_i+\mathbf{h}}(\omega_2),\ldots,J^{(x)}_{\mathbf{s}_i,\mathbf{s}_i+\mathbf{h}}(\omega_M)\},$$

which is distributed asymptotically as complex multi-variate normal with mean zero and variance covariance matrix with diagonal elements

$$\left[g_0^{\|\mathbf{h}\|}(\omega_1,\boldsymbol{\psi}), g_0^{\|\mathbf{h}\|}(\omega_2,\boldsymbol{\psi}),\ldots,g_0{}^{\|\mathbf{h}\|}(\omega_M,\boldsymbol{\psi})\right].$$

We note that off-diagonal elements of the covariance matrix are zero. Proceeding as in Subba Rao *et al.* (2014), we can show that the log likelihood function $l(\psi/J_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega))$ is proportional to

$$Q^{(\mathbf{h})}_{n,i}(\boldsymbol{\psi})=\sum_{k=1}^{M}\left[\ln g_0{}^{\|\mathbf{h}\|}(\omega_k,\boldsymbol{\psi})+\frac{I^{x}_{\mathbf{s}_i,\mathbf{s}_{i+\mathbf{h}}}(\omega_k)}{g_0{}^{(\mathbf{h})}(\omega_k,\psi)}\right].$$

Now consider all the locations $(\mathbf{s}_i,\mathbf{s}_i+\mathbf{h}); i=1,2,\ldots,m$ belonging to the set $\mathrm{N}(\mathbf{h})$. Then we have the pooled criterion

$$Q_{n,\mathrm{N}(\mathbf{h})}(\boldsymbol{\psi})=\frac{1}{|\mathrm{N}(\mathbf{h})|}\sum_{(\mathbf{s}_i,\mathbf{s}_i\in\mathrm{N}(\mathbf{h}))}Q^{(\mathbf{h})}_{n,i}(\boldsymbol{\psi}). \tag{16}$$

Suppose we have H spatial distances $\{\mathbf{h}(l); l=1,2,\ldots,H\}$ for which the intrinsic stationarity condition is satisfied, and then we can define an overall measure for minimization,

$$Q_n(\boldsymbol{\psi})=\frac{1}{H}\sum Q_{n,\mathrm{N}(\mathbf{h}_l)}(\boldsymbol{\psi}). \tag{17}$$

We minimize (17) with respect to $\boldsymbol{\psi}$. The asymptotic normality of the estimator of $\boldsymbol{\psi}$ can be proved using the methodology described by Subba Rao *et al.* (2014). For large n, we can show

$$\sqrt{n}(\tilde{\boldsymbol{\psi}}-\boldsymbol{\psi})\xrightarrow{D}N(0,\left[\nabla^2 Q_n(\boldsymbol{\psi})\right]^{-1}V\ \left[\nabla^2 Q_n(\boldsymbol{\psi})\right]\ ),$$

where $V=\lim_{n\to\infty}\mathrm{Var}\left[\frac{1}{\sqrt{n}}\nabla Q_{n,}(\boldsymbol{\psi})\right]$, and $\nabla Q_n(\boldsymbol{\psi})$ is a Jacobian vector of first-order partial derivatives, and $\left[\nabla^2 Q_n(\boldsymbol{\psi})\right]$ is a Hessian matrix of second-order partial derivatives.

9. TEST FOR INDEPENDENCE OF m SPATIAL TIME SERIES

So far, we have considered the analysis of spatio-temporal data using various frequency domain methods. We assumed that there is a second-order dependence in space and time. It is important to test for independence over space and time before modelling the data. Henebry (1995) proposed a test statistic for testing spatio-temporal independence; and the test proposed is as an extension of Moran's test. In their book, Cressie and Wikle (2011) briefly discussed the test. In this section, we propose a test for spatial independence using the DFTs, and the test is based on the test proposed by Wahba (1971), which is an extension of the classical test for independence used in multi-variate analysis. Here we briefly describe the test. Let

$$\underline{Y}'_t=(Y_t(\mathbf{s}_1),Y_t(\mathbf{s}_2),\ldots,Y_t(\mathbf{s}_m)).$$

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We say that the multi-variate time series $\{\underline{Y}_t\}$ is second-order stationary if (Brockwell and Davis, 1987)

1. $E(\underline{Y}_t) = \underline{\mu}$;
2. $E(\underline{Y}_t - \underline{\mu})(\underline{Y}_{t+p} - \underline{\mu})^{'} = \underline{\Gamma}(p)$, where

$$\begin{aligned}\underline{\mu}^{'} &= (\mu_1, \mu_2, \dots, \mu_m),\\ \Gamma(p) &= (\sigma_{ij}(p)),\\ \sigma_{ij}(p) &= E\left(Y_t(\mathbf{s}_i) - \mu_i\right)\left(Y_{t+p}(\mathbf{s}_j) - \mu_j\right), (i, j = 1, 2, \dots, m),\\ \sigma_{ij}(p) &= \sigma_{ji}(-p).\end{aligned}$$

Here we are assuming that the spatio-temporal data are temporally stationary only and that no assumption of spatial stationarity is assumed. We assume further that $\underline{Y}_t$ is Gaussian. Define the complex valued random vector

$$\underline{J}'(\omega_k) = (J^y_{\mathbf{s}_1}(\omega_k), J^y_{\mathbf{s}_2}(\omega_k), \dots, J^y_{\mathbf{s}_m}(\omega_k)),$$

where $J^y_{\mathbf{s}_i}(\omega_k)$ is the DFT of the time-series data $\{Y_t(\mathbf{s}_i)\}$, and $\omega_k = \frac{2\pi k}{n}$, $(k =, 1, \dots, \left[\frac{n}{2}\right])$. We know that the random vector $\underline{J}(\omega_k)$ is distributed as complex normal with mean $\mathbf{0}$ and variance covariance matrix $\underline{F}(\omega_k)$, where $\underline{F}(\omega_k) = \left[E(J^y_{\mathbf{s}_i}(\omega_k)J^{y*}_{\mathbf{s}_j}(\omega_k)\right]$. We note that $\underline{F}(\omega_k)$ is a Hermitian matrix, with elements

$$f_{\mathbf{s}_i,\mathbf{s}_j}(\omega_k) = E(J^y_{\mathbf{s}_i}(\omega_k)J^{y*}_{\mathbf{s}_j}(\omega_k)) = f_{\mathbf{s}_j,\mathbf{s}_i}(-\omega_k).$$

In the preceding result, $f_{\mathbf{s}_i,\mathbf{s}_i}(\omega_k)$ is the second-order spectral density function of the process $\{Y_t(\mathbf{s}_i)\}$, and $f_{\mathbf{s}_i,\mathbf{s}_j}(\omega_k)$ is the cross-spectral density function of the process $\{Y_t(\mathbf{s}_i)\}$ and $\{Y_t(\mathbf{s}_j)\}$. The cross-spectral density function is usually a complex valued function.

If we assume that the spatio-temporal process $\{Y_t(\mathbf{s})\}$ is stationary in space and time, and further assume that the process is isotropic in space, then

$$\begin{aligned}f_{\mathbf{s}_i,\mathbf{s}_i}(\omega) &= f_0(\omega),\\ f_{\mathbf{s}_i,\mathbf{s}_j}(\omega) &= f_{||\mathbf{s}_i - \mathbf{s}_j||}(\omega).\end{aligned}$$

In this case, the matrix $\underline{F}(\omega)$ is real and symmetric, and all the diagonal elements are equal to $f_0(\omega)$.

As pointed out earlier, for testing spatial independence, we do not need the assumption of spatial stationarity. Subsequently, we assume that the process is Gaussian. Under the null hypothesis that the spatial process is spatially independent, the spectral matrix $F(\omega)$ is a diagonal matrix for all $|\omega| \le \pi$. For constructing the test, we proceed as in Wahba (1971). Consider the DFTs defined earlier. For each location s_i, let the Fourier transform be given by $(J^y_{s_i}(\omega_l))$, where $\omega_l = \frac{2\pi j_l}{n}$, $j_l = (l-1)(2k+1) + (k+1)$; $l = 1, 2, \dots, M_1$, where M_1 is chosen such that $2(k+1)M_1 = \frac{n-1}{2}$. (Here we assume that the number of observations n is odd.) As in Wahba (1971), we define the cross-spectral estimator of $f_{s_i,s_j}(\omega)$ by

$$\hat{f}_{s_i,s_j}(\omega_l) = \frac{1}{2k+1}\sum_{j_1=-k}^{k} I_{i,j}(\omega_l + \frac{2\pi j_1}{n}), \quad (l = 1, 2, \dots, M_1),$$

where the cross-periodogram $I_{ij}(\omega_l) = J^y_{s_i}(\omega_l)J^{y*}_{\mathbf{s}_j}(\omega_l)$.

Let $\hat{F}(\omega_l) = (\hat{f}_{s_i,s_j}(\omega_l)) \quad (l = 1, 2, \dots, M_1)$.

We note that the random matrices $\hat{F}(\omega_l); l = 1, 2, \dots, M_1$, for large k, are approximately distributed as random matrices $\tilde{F}(\omega_l)$, $(l = 1, 2, \dots, M)$, which are distributed as complex Wishart, usually denoted by $W_c(F, m, 2k+1)$.

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Wahba (1971) has shown that the likelihood ratio test for testing the null hypothesis that the matrices $F(w_l)$ are diagonal for all $\{\omega_l\}$ leads to the test statistic, for each w_l,

$$\tilde{\boldsymbol{\lambda}}_l = \frac{|\tilde{F}(\omega_l)|}{\prod_{j=1}^{m} \tilde{f}_{\mathbf{s}_j,\mathbf{s}_j}(\omega_l)} \quad (l = 1, 2, \ldots, M_1),$$

and the overall test statistic to consider is $\boldsymbol{\Lambda} = -\frac{1}{M_1}\sum \ln \tilde{\boldsymbol{\lambda}}_l$. For large k and M_1, under the null hypothesis, the statistic $\boldsymbol{\Lambda}$ is asymptotically distributed as normal with mean

$$E(\boldsymbol{\Lambda}) = \sum_{j=1}^{m-1} \frac{m-j}{k'-j}$$

and variance

$$\mathrm{Var}(\boldsymbol{\Lambda}) = \frac{1}{M_1}\sum_{j=1}^{m-1} \frac{m-j}{(k'-j)^2},$$

where $k' = 2k + 1$. Under the null hypotheses of spatial independence, for large k and M, the statistic $S = \frac{\boldsymbol{\Lambda} - E(\boldsymbol{\Lambda})}{\sqrt{\mathrm{Var}(\boldsymbol{\Lambda})}}$ is distributed as standard normal. We note that if for each $\mathbf{s}_i$, $\{Y_t(\mathbf{s}_i)\}$ is a Gaussian white noise, then the spectral density function is given by $f_{\mathbf{s}_i,\mathbf{s}_i}(\omega) = \frac{\sigma_{s_i}{}^2}{2\pi}$, where $\sigma_{\mathbf{s}_i}{}^2$ is the variance of the white noise. If the null hypothesis is both spatially and temporally independent, then the diagonal elements of the matrix $F(\omega_l)$ will be proportional to $(\sigma_{\mathbf{s}_1}{}^2, \sigma_{\mathbf{s}_2}{}^2, \sigma_{\mathbf{s}_3}{}^2, \ldots, \sigma_{\mathbf{s}_m}{}^2)$, and all off-diagonal elements will be zero.

APPENDIX: DISCRETE FOURIER TRANSFORMS

We will briefly summarize some results related to the discrete Fourier transforms; for further details, we refer to Subba Rao and Terdik (2015), Brillinger (2001) and Giraitis *et al.* (2012).

Let $\{Y_t(\mathbf{s})\}$, where $\{\mathbf{s} \in \mathbb{R}^d; t \in Z\}$ denote a zero-mean second-order spatially, temporally stationary process with spectral representation

$$Y_t(\mathbf{s}) = \int_{-\infty}^{\infty}\int_{-\pi}^{\pi} e^{i(\mathbf{s}\cdot\boldsymbol{\lambda} + t\omega)} \mathrm{d}Z_y(\boldsymbol{\lambda}, \omega), \tag{A1}$$

and let $\{Y_t(\mathbf{s_i})); i = 1, 2, \ldots, m; t = 1, 2, \ldots, n\}$ be a sample from the process $\{Y_t(\mathbf{s})\}$. We note that $Z_y(\boldsymbol{\lambda}, \omega)$ is a zero-mean complex valued function with orthogonal increments and

$$E[\mathrm{d}Z_y(\boldsymbol{\lambda}, \omega)] = 0,$$
$$E|\mathrm{d}Z_y(\boldsymbol{\lambda}, \omega)|^2 = \mathrm{d}F_y(\boldsymbol{\lambda}, \omega),$$

where $\mathrm{d}F_y(\boldsymbol{\lambda}, \omega)$ is a spectral measure. Let $\mathrm{d}F_y(\boldsymbol{\lambda}, \omega) = f_y(\boldsymbol{\lambda}, \omega)\mathrm{d}\boldsymbol{\lambda}\mathrm{d}\omega$, where $f_y(\boldsymbol{\lambda}, \omega)$ is the spatio-temporal spectral density function of the process $\{Y_t(\mathbf{s})\}$. Define the discrete Fourier transform

$$J_{\mathbf{s}}^y(\omega) = \frac{1}{\sqrt{2\pi n}} \sum_{t=1}^{n} Y_t(\mathbf{s}) e^{it\omega}, |\omega| \le \pi. \tag{A2}$$

Proposition 5. Let the spectral representation of the process $\{Y_t(\mathbf{s}_i)\}$ be given by (A1), and let $J_{\mathbf{s}}(\omega)$ be the DFT of the sample $\{Y_t(\mathbf{s}); t = 1, 2, \ldots, n\}$. Then we have

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1. $Y_t(\mathbf{s}) = \sqrt{\frac{n}{2\pi}} \int J_{\mathbf{s}}^{y}(\omega) e^{it\omega} \mathrm{d}\omega$.
2. $J_{\mathbf{s}}^{y}(\omega) \approx \int e^{is\boldsymbol{\lambda}} \sqrt{\frac{n}{2\pi}} \mathrm{d}Z_y(\boldsymbol{\lambda}, \omega)$.

Proof

By substitution and using the properties of Dirac delta function, one can show (2). (1) follows by inversion of (A2). For details, refer to Subba Rao and Terdik (2015). □

Let $I_{\mathbf{s}}^{y}(\omega_k) = |J_{\mathbf{s}}^{y}(\omega_k)|^2$ be the periodogram. The following results are well known (Priestley, 1981; Brillinger, 2001):

1. $E(I_{\mathbf{s}}^{y}(\omega_k)) = g_{\mathbf{s}}^{y}(\omega_k) + O(n^{-1})$.
2. $\mathrm{Var}(I_{\mathbf{s}}^{y}(\omega_k)) = g_{\mathbf{s}}^{y2}(\omega_k) + O(n^{-1}), \quad \omega_k \neq 0, \pi$.
3. $\mathrm{Cov}(I_{\mathbf{s}}^{y}(\omega_k), I_{\mathbf{s}}^{y}(\omega_l)) = O(n^{-1}) \quad \text{if} \quad \omega_k + \omega_l \neq 0 \pmod{2\pi}$. In view of spatial stationarity, $g_{\mathbf{s}}^{y}(\omega) = g_{0}^{y}(\omega)$ for all $\mathbf{s}$, and

$$g_{\mathbf{s}}^{y}(\omega) = \frac{1}{2\pi} \sum_{k} \mathrm{Cov}(Y_t(\mathbf{s}), Y_{t+k}(\mathbf{s})) e^{-i\omega k}, |\omega| \leq \pi.$$

4. $\mathrm{Cov}(J_{\mathbf{s}_i}^{y}(\omega_k), J_{\mathbf{s}_j}^{y}(\omega_k)) = O(n^{-1}), \quad \text{if} \quad \omega_k + \omega_l \neq 0 \pmod{2\pi}$.
5. $\mathrm{Cov}(J_{\mathbf{s}_i}^{y}(\omega_k), J_{\mathbf{s}_j}^{y}(\omega_k)) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} c(\mathbf{s}_i - \mathbf{s}_j, n) e^{-in\omega_k} = g_{\mathbf{s}_i - \mathbf{s}_j}(\omega_k) + O(n^{-1})$. If the process is isotropic, then the spectral density function $g_{\mathbf{s}_i - \mathbf{s}_j}(\omega_k) = g_{||\mathbf{s}_i - \mathbf{s}_j||}(\omega_k)$, which is a real-valued function.

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REFERENCES

Brillinger DR. 2001. *Time Series; Data Analysis and Theory*, Reprint of the 1981 edition. Philadelphia, PA: Society for Industrial and Applied Mathematics (SIAM).

Brockwell PJ, Davis RA. 1987. *Time Series: Theory and Methods*. New York: Springer-Verlag.

Cressie N. 1993. *Statistics for Spatial Data*, Wiley Series in Probability and Mathematical Statistics. New York: Wiley.

Cressie N, Huang HC. 1999. Classes of nonseparable, spatio-temporal stationary covariance functions. *Journal of the American Statistical Association* **94**(448): 1330–1339.

Cressie N, Wikle CK. 2011. *Statistics for Spatio-temporal Data*: Wiley Series in Probability and Statistics.

Deo RS, Chen WW. 2000. On the integral of the squared periodogram. *Stochastic processes and their applications* **85**(1): 159–176.

Giraitis L, Koul HL, Surgailis D. 2012. *Large Sample Inference for Long Memory Processes*, Vol. 201/2: World Scientific.

Gneiting T. 2002. Nonseparable, stationary covariance functions for space–time data. *Journal of the American Statistical Association* **97**(458): 590–600.

Gneiting T, Sasvári Z, Schlather M. 2001. Analogies and correspondences between variograms and covariance functions. *Advances in Applied Probability* **33**(3): 617–630.

Gringarten E, Deutsch CV. 2001. Teacher's aide variogram interpretation and modeling. *Mathematical Geology* **33** (4): 507–534.
Hannan EJ. 1973. The asymptotic theory of linear time series models. *Journal Applied Probability* **10**: 130–145.
Henebry GM. 1995. Spatial model error analysis using autocorrelation indices. *Ecological Modelling* **82**(1): 75–91.
Huang C, Hsing T, Cressie N. 2011. Nonparametric estimation of the variogram and its spectrum. *Biometrika* **98**(4): 775–789.
Li B, Genton MG, Sherman M. 2007. A nonparametric assessment of properties of space–time covariance functions. *Journal of the American Statistical Association* **102**(478): 736–744.
Ma C. 2005. Spatio-temporal variograms and covariance models. *Advances in Applied Probability* **37**(3): 706–725.
Matheron G. 1963. Principles of geostatistics. *Economic Geology* **58**(8): 1246–1266.
Priestley MB. 1981. *Spectral Analysis and Time Series*. New York: Academic Press.
Schabenberger O, Gotway CA. 2005. *Statistical Methods for Spatial Data Analysis*: CRC Press.
Sherman M. 2011. *Spatial Statistics and Spatio-temporal Data Covariance Function and Directional Statistics*. UK: John Wiley.
Stein ML. 2012. *Interpolation of Spatial Data: Some Theory for Kriging*: Springer Science & Business Media.
Stein ML. 2005a. Space–time covariance functions. *Journal of the American Statistical Association* **100**(469): 310–321.
Stein ML. 2005b. Statistical methods for regular monitoring data. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* **67**(5): 667–687.
Subba Rao T, Terdik Gy. 2015. A space–time covariance function for spatio-temporal random processes and spatio-temporal prediction (Kriging). *ArXiv e-prints, 311.1981v2 [math.ST]*. (submited for publication).
Subba Rao T, Das S, Boshnakov G. 2014. A frequency domain approach for the estimation of parameters of spatio-temporal random processes. *Journal of Time Series Analysis* **35**: 357–377.
Taniguchi M. 1980. On estimation of the integrals of certain functions of special density. *J. Appl.prob.* **17**: 73–83.
Wahba G. 1971. Some tests of independence for stationary multivariate time series. *Journal of the Royal Statistical Society. Series B (Methodological)* **33**(1): 153–166.
Yaglom AM. 1987. *Correlation Theory of Stationary Related Random Functions*. New York: Springer-Verlag.
Yu K, Mateu J, Porcu E. 2007. A kernel-based method for non parametric estimation of variogram. *Statistica Neerlandica* **61**(2): 173–197.

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